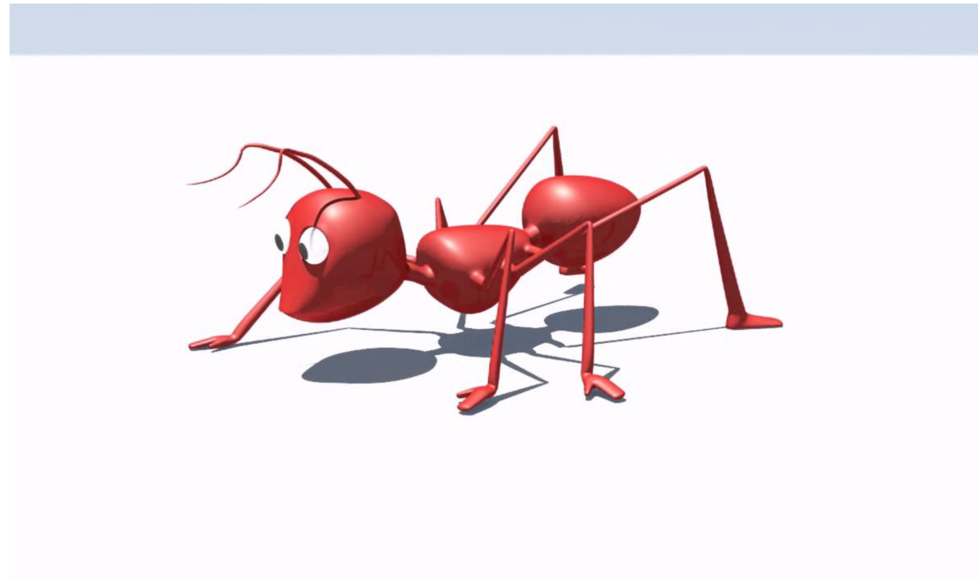
SARSI 2016
First Week Lectures
Math – Kim Whittlesey

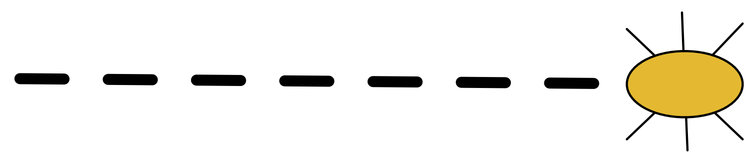
Lecture 1
Non-Euclidean Geometry
الهندسة الإقليدية غير

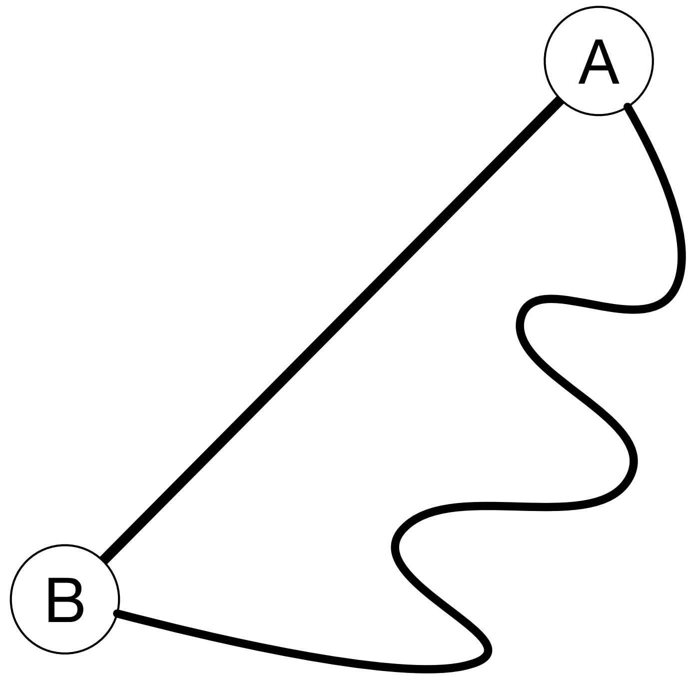
In the Euclidean plane

1. What is a straight line?
2. What does parallel mean?

If a bug walks on the plane and does not turn left or right, then it goes in a straight line.

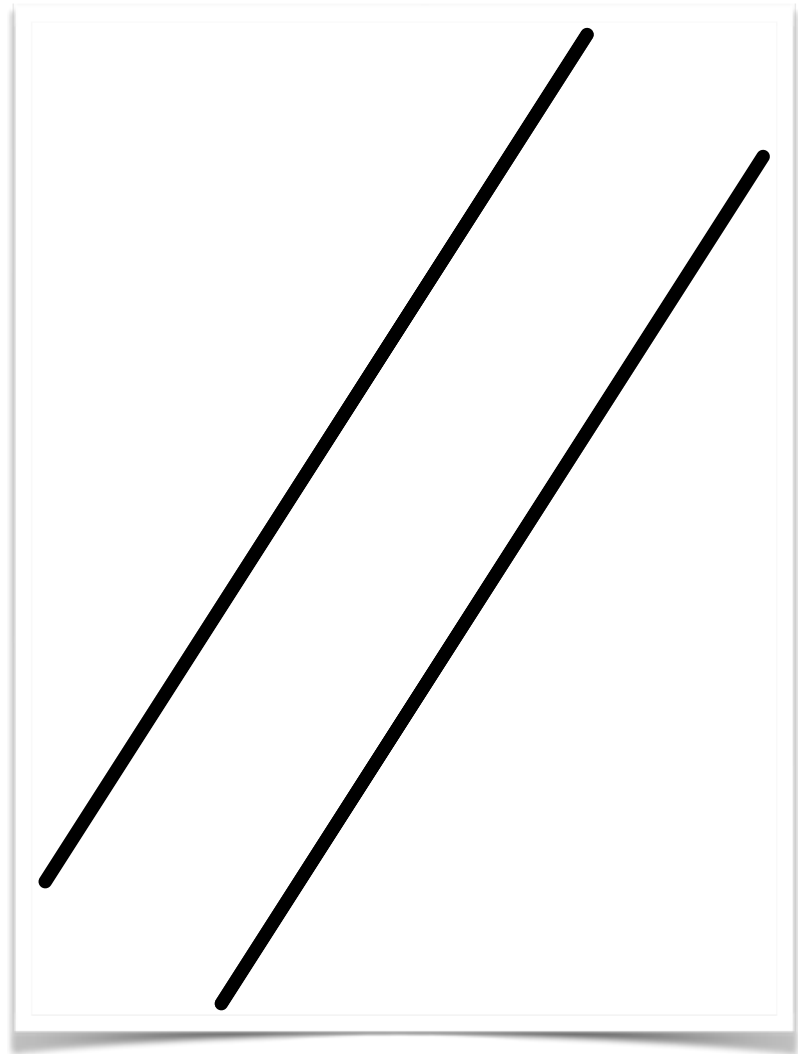






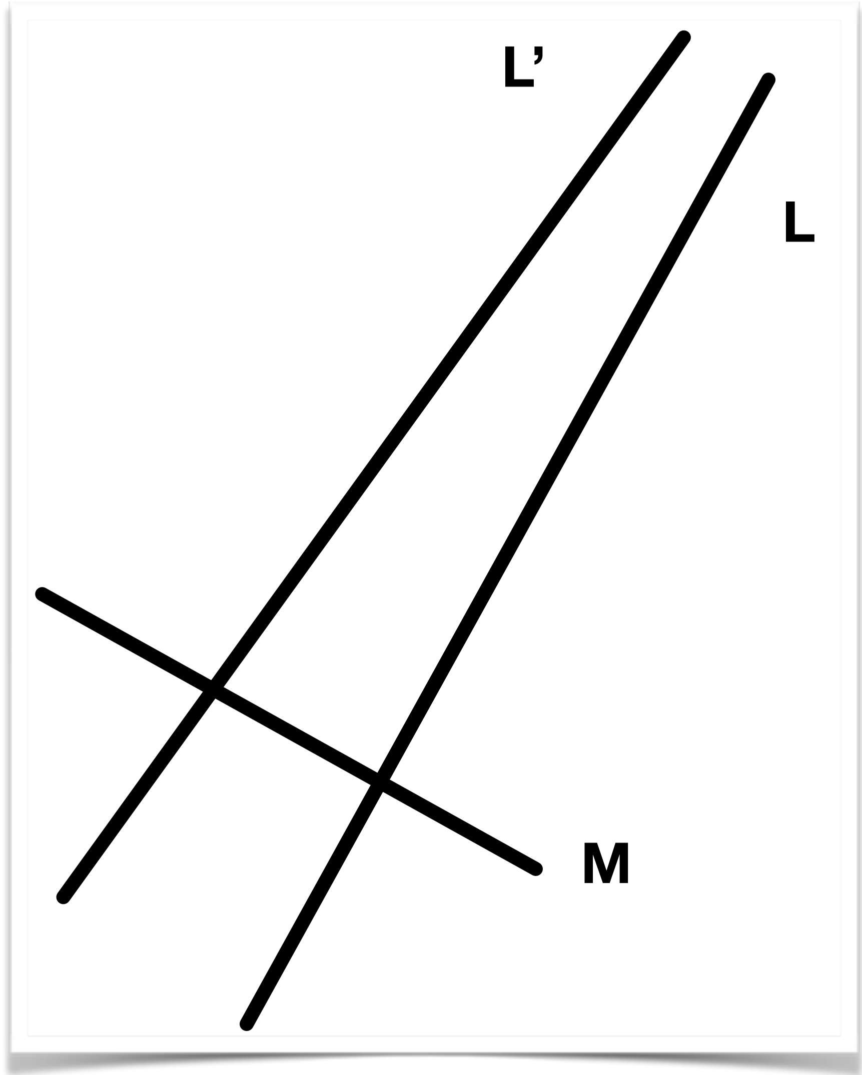
Also, the shortest curve connecting two points is on a straight line.

If two lines do not intersect, we say they are parallel.



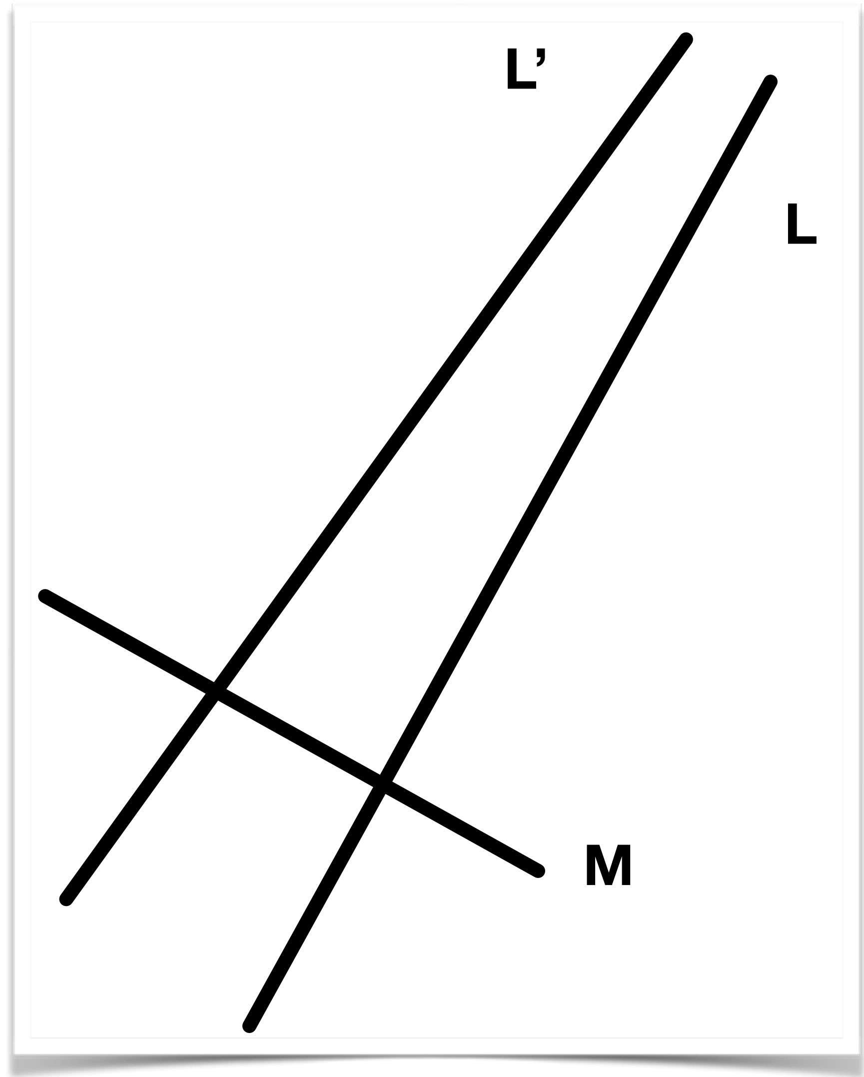
Suppose lines L and L′ meet line M at different angles.

Do L and L′ meet?



Yes.

This is Euclid's 5th Axiom, which we'll talk more about later.

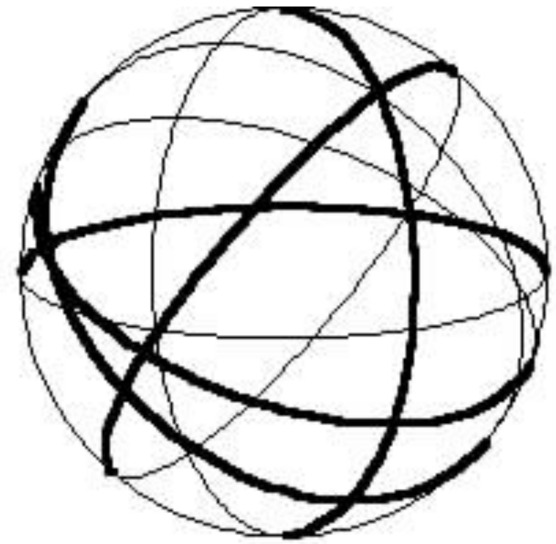


The Sphere

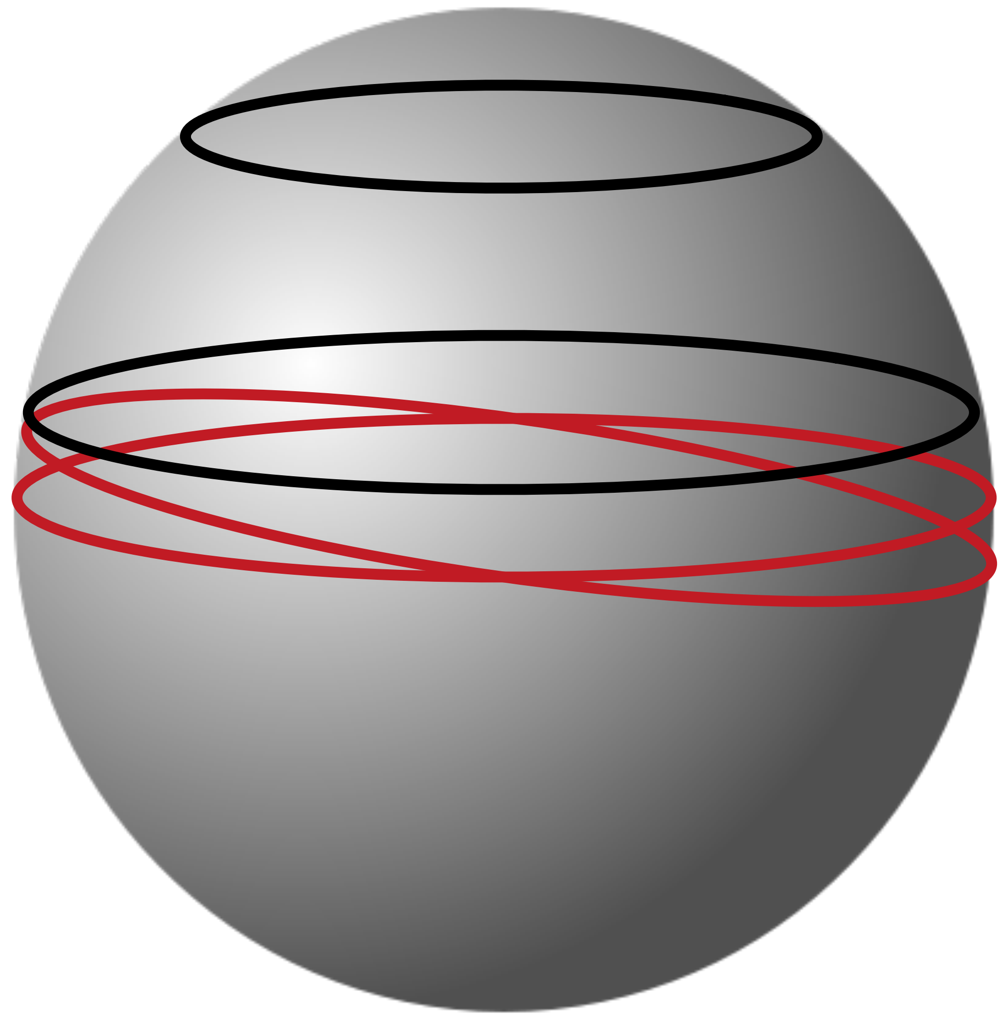
On the Sphere:

1. What is a straight line?
2. Can straight lines be parallel?

On a sphere, straight lines are great circles.

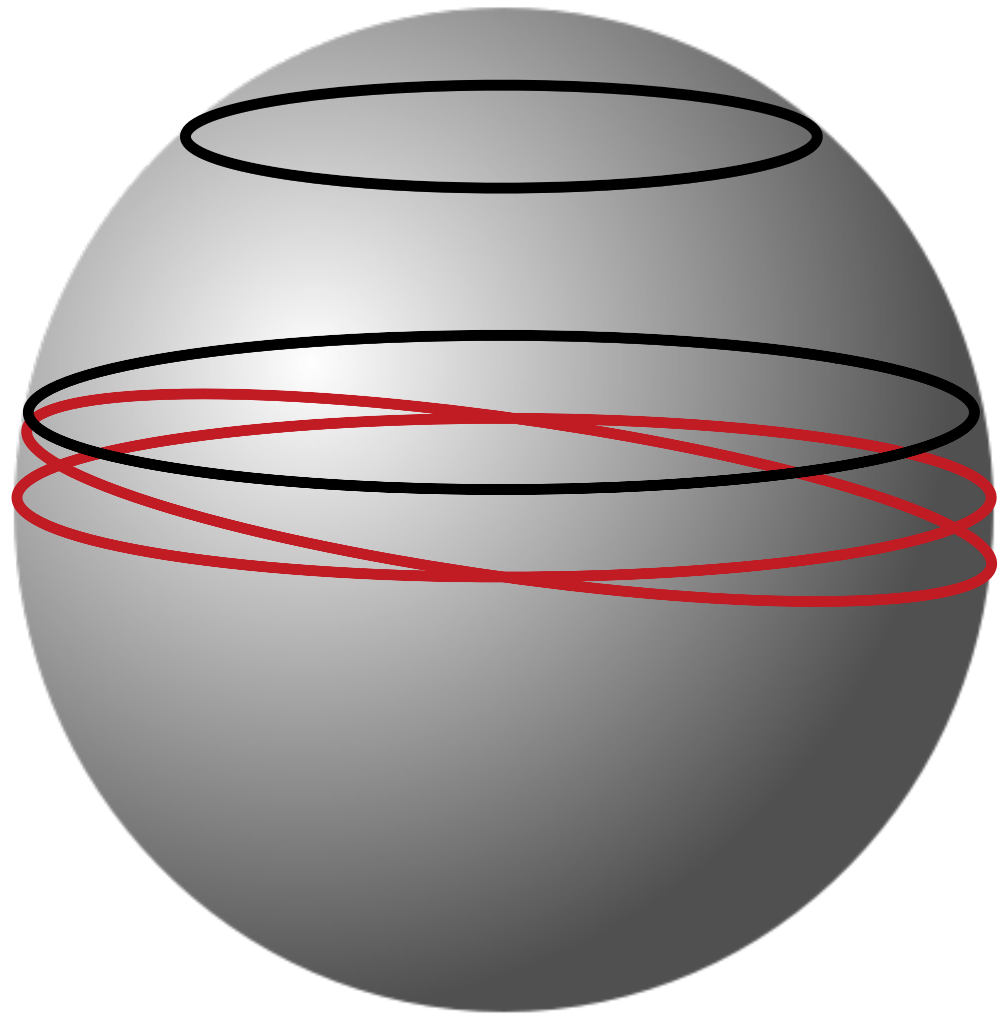


Which of these curves are straight?

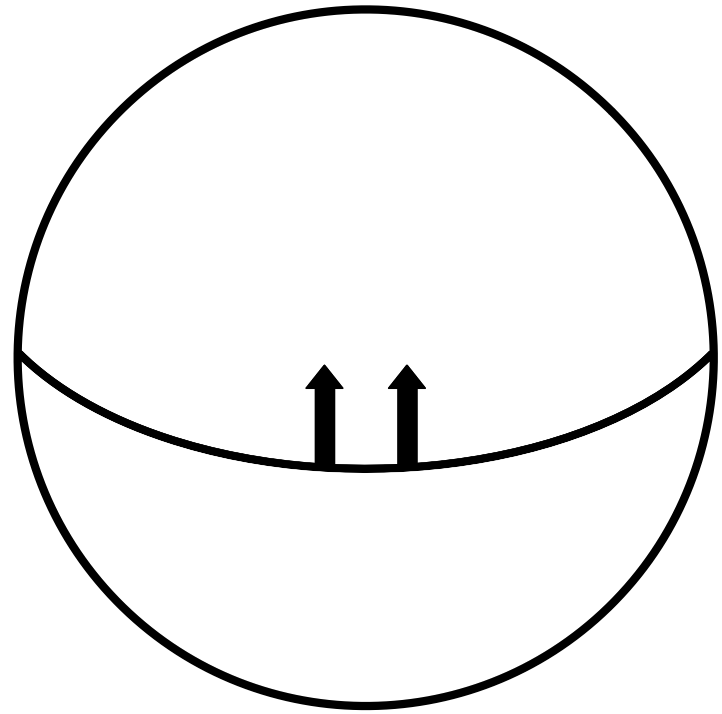


Only the red curves here are straight.

Curves along the smaller circles are not straight.

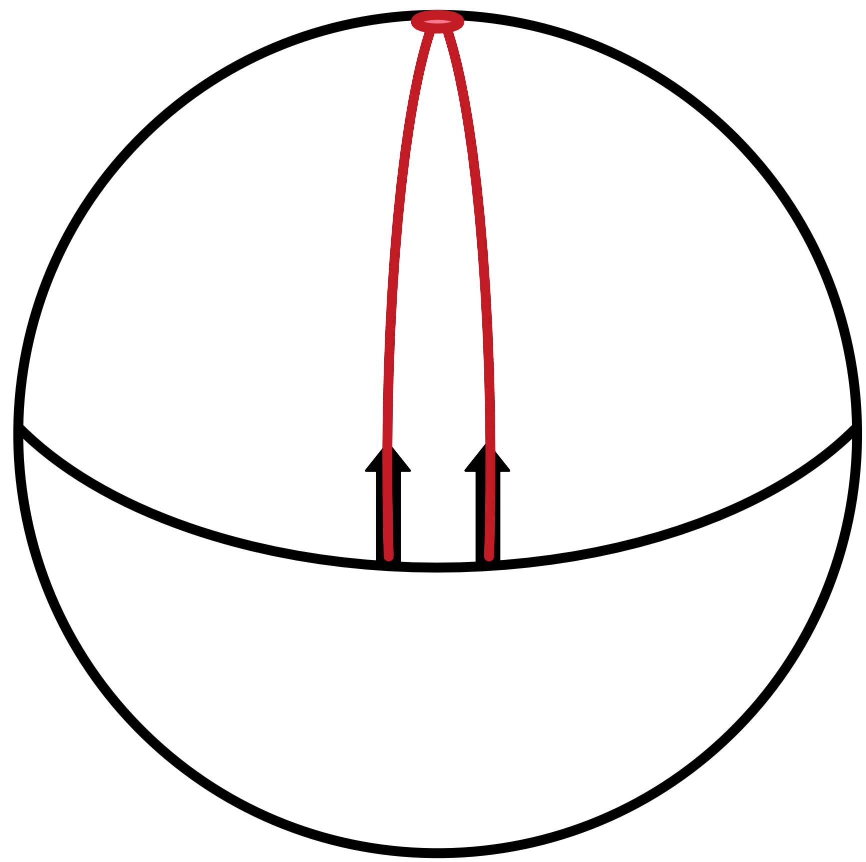


What happens if two people at the equator both start going north along straight lines?

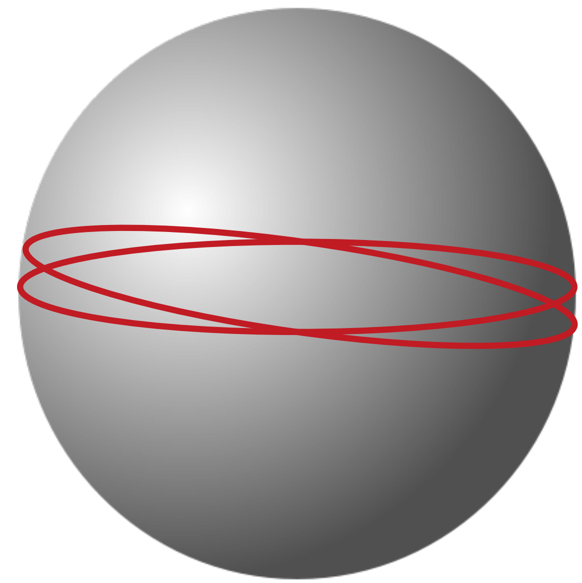


What happens if two people at the equator both start going north along straight lines?

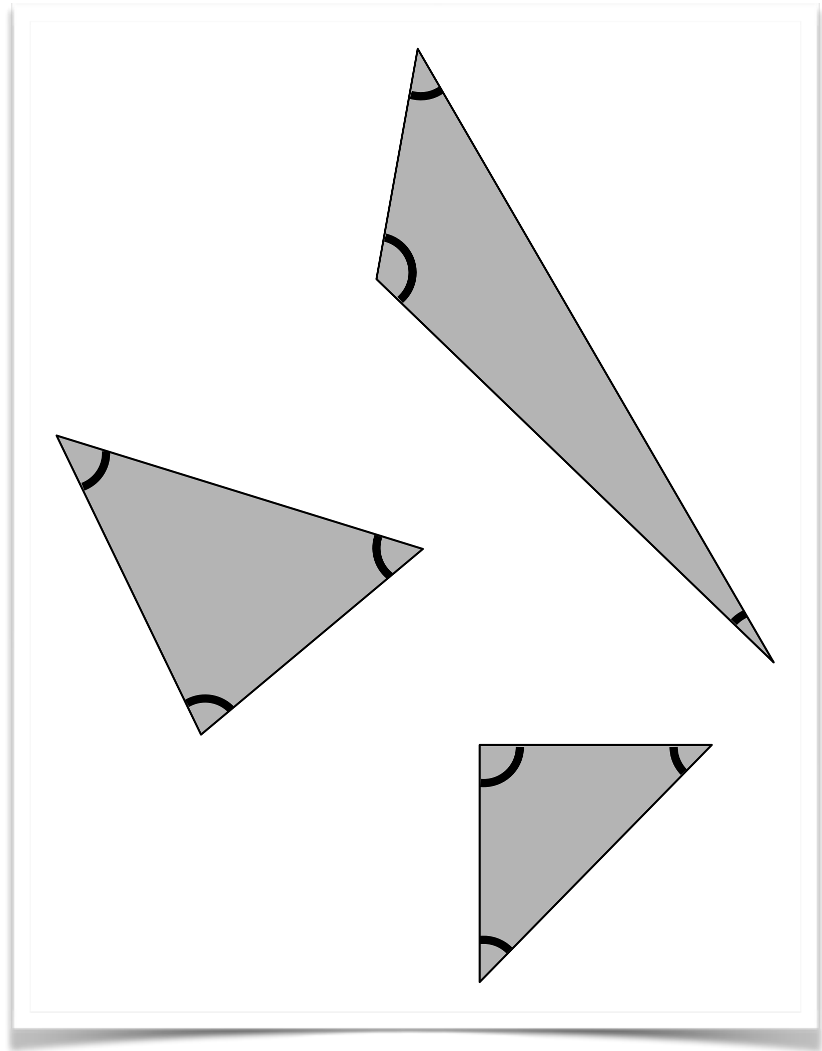
They'll meet at the north pole.



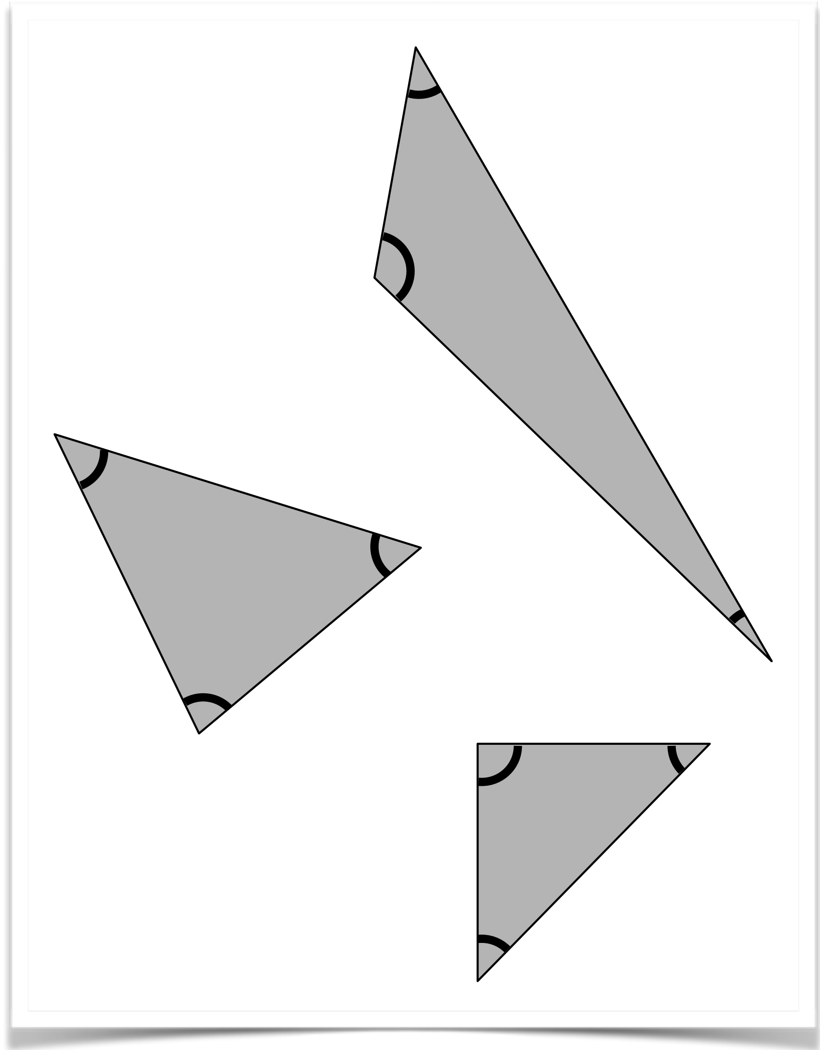
Straight lines (great circles) on the sphere are never parallel.



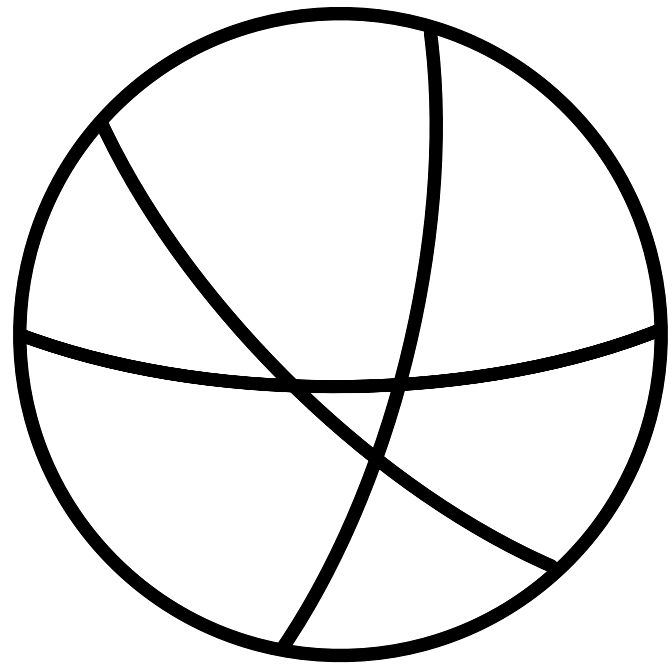
On the Euclidean plane, the sum of the interior angles of a triangle is always . . .



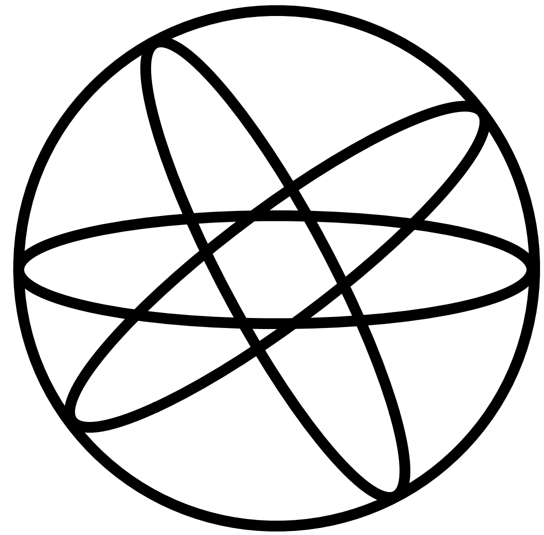
On the Euclidean plane, the sum of the interior angles of a triangle is always 180°.



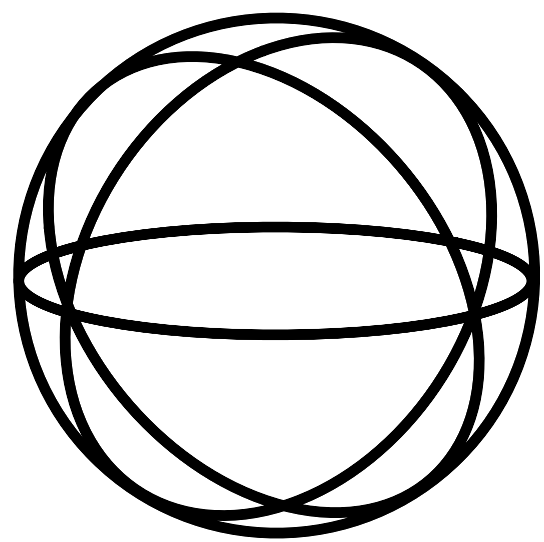
On a sphere, what can the sum of the interior angles be?



Small triangles have angle sum just over 180°.

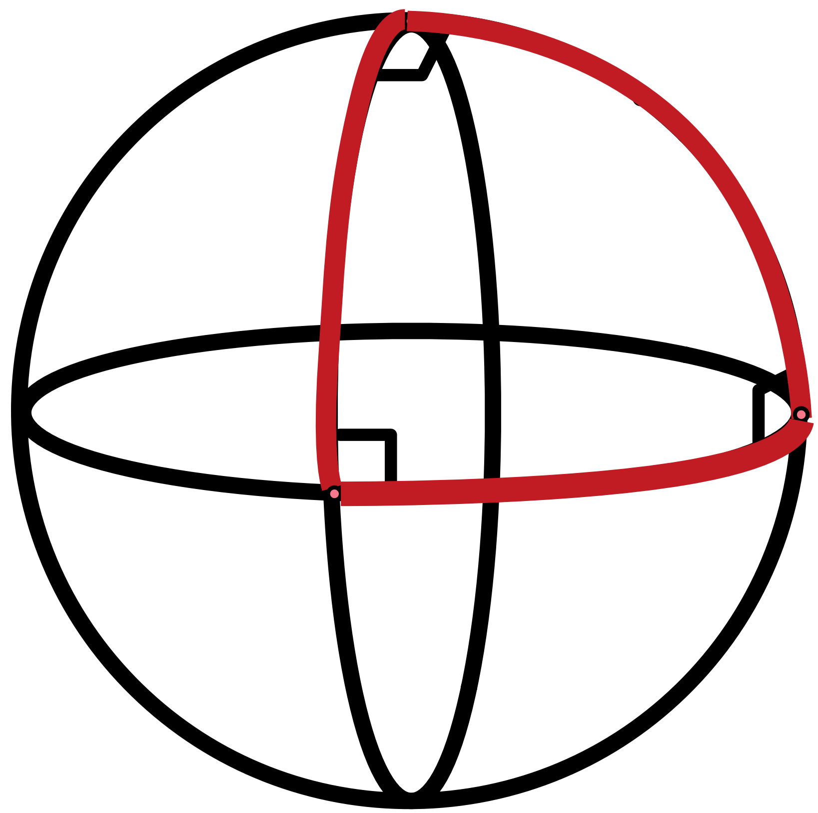


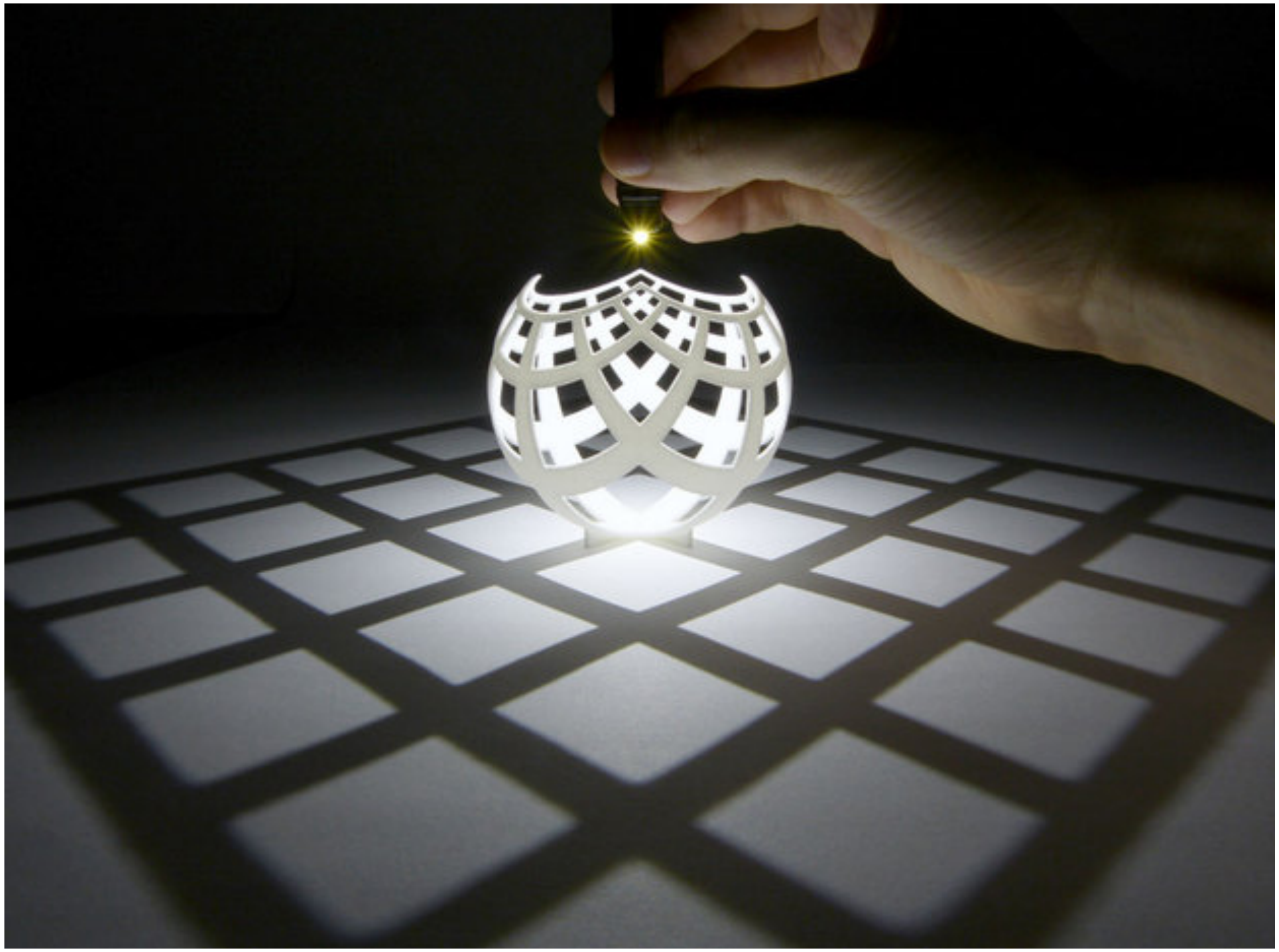
Big triangles can have much larger angle sum.



Problem: Find a triangle on the sphere with angle sum 270°.

You can make a triangle on the sphere with 3 right angles.





sculpture by Henry Segerman

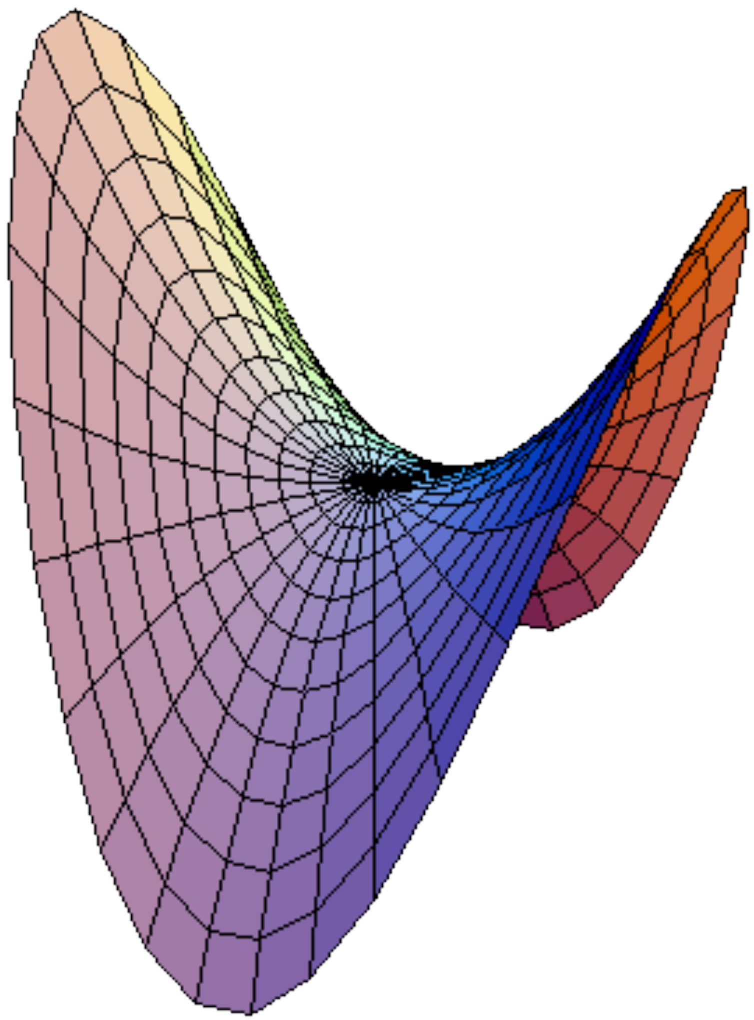
If you try to flatten the sphere to the plane, you have to distort shapes.

The Hyperbolic plane

knitting by Daina Taimina, photo by Steve Rowell

(a bit of) the Hyperbolic Plane

a hyperbolic leaf



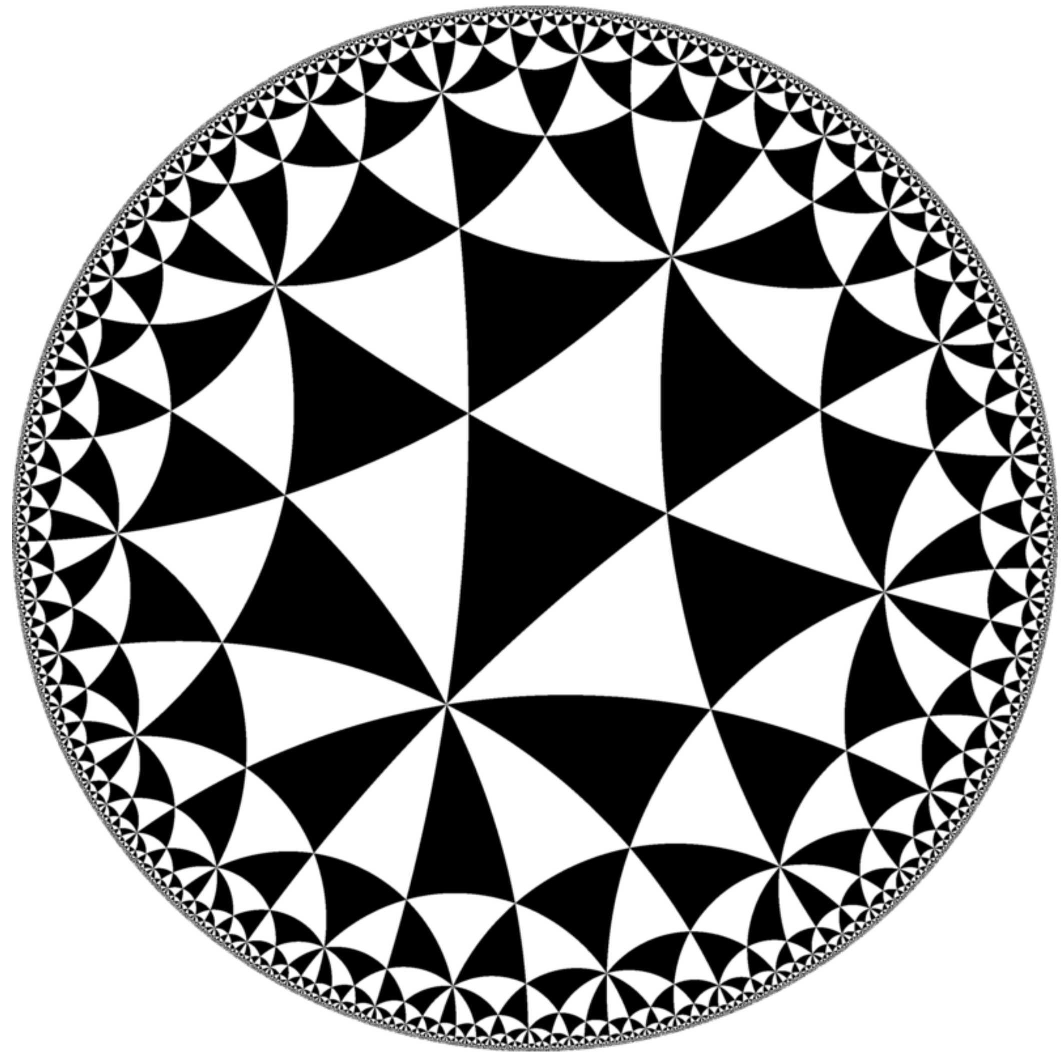
Locally, every point looks like it is on a saddle.

We’ll have to distort the space a bit to draw it in the plane.

We’ll talk about two models.

Our first model of hyperbolic space is the Poincaré disk.

Here, the boundary circle is at infinity.

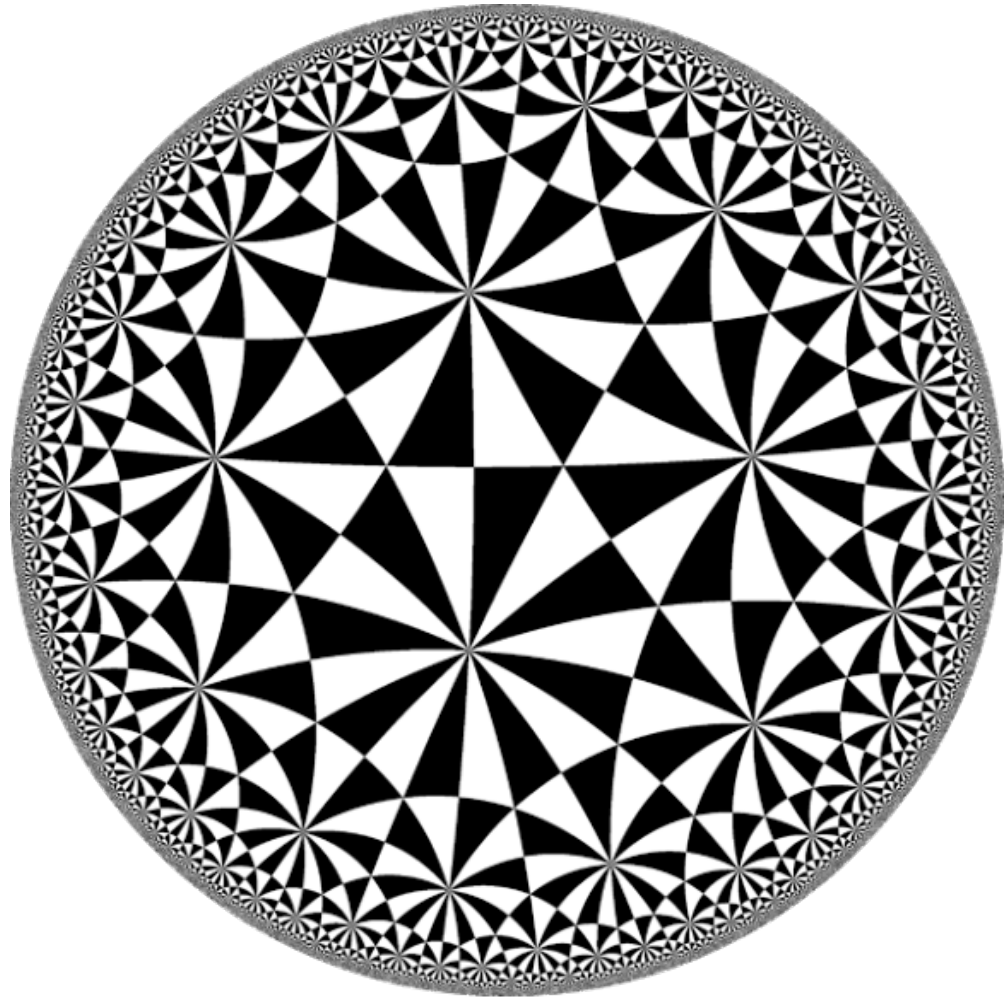


The same distance looks smaller and smaller as you go out to the boundary.

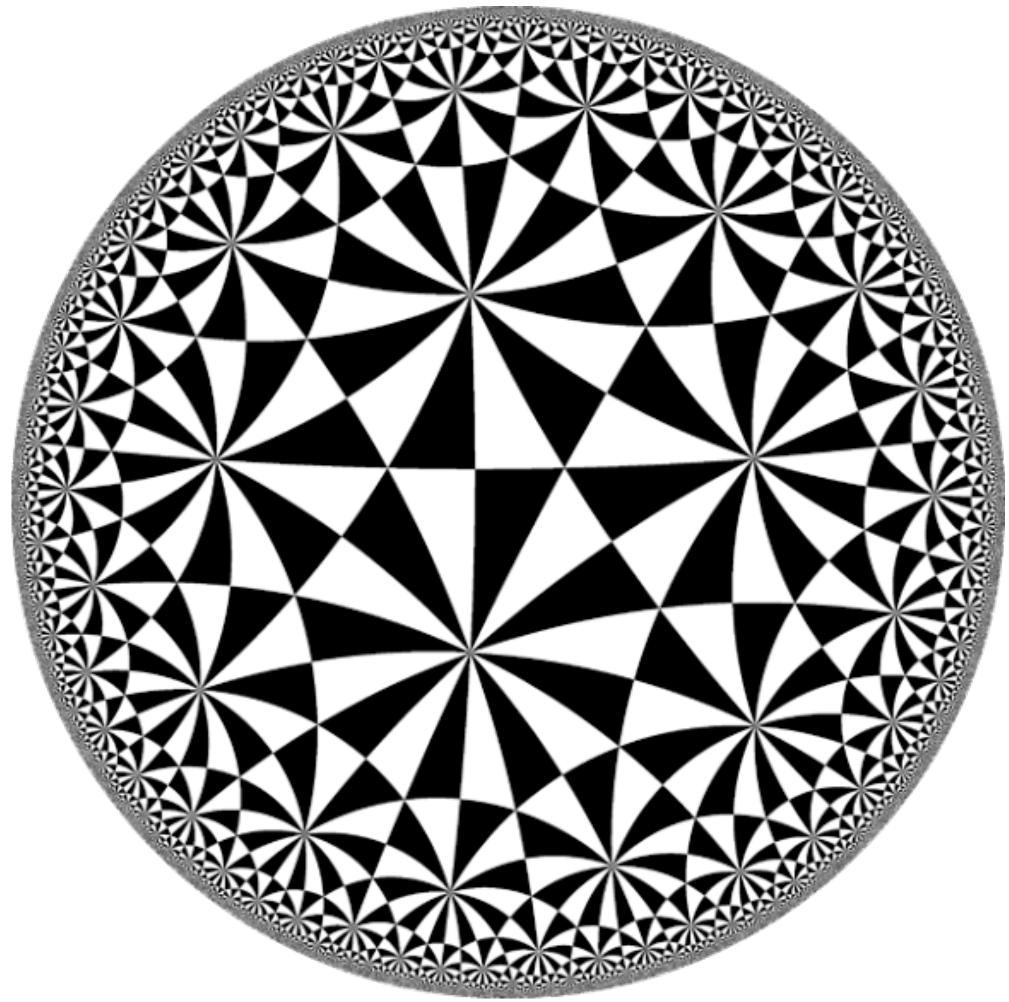
Picture credit: MC Escher, Josh Leyes

So distances
are wrong.

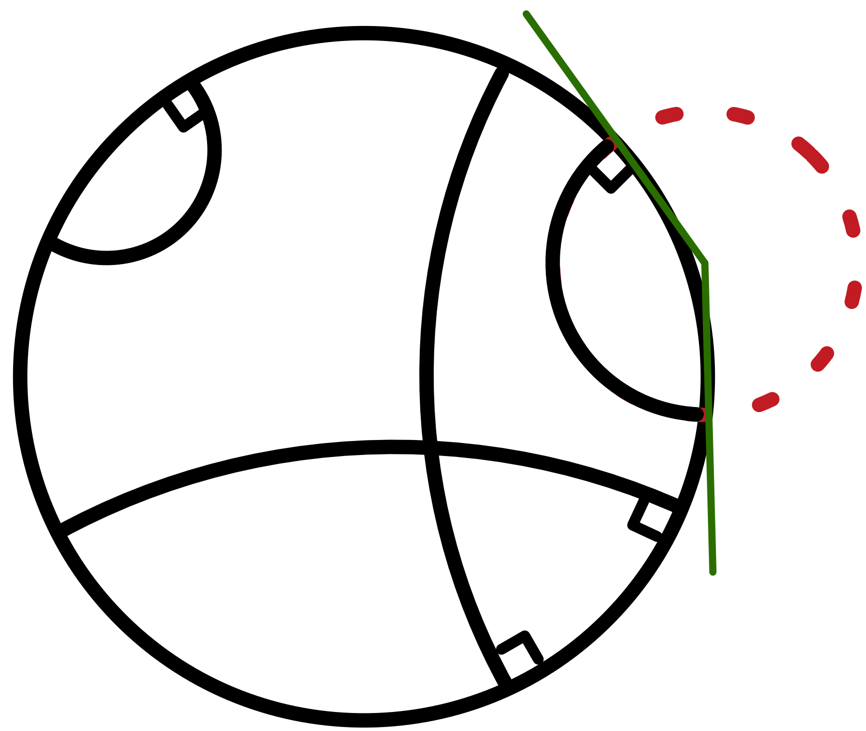
But the angles
are correct.



What are the straight lines in the Poincaré disk?

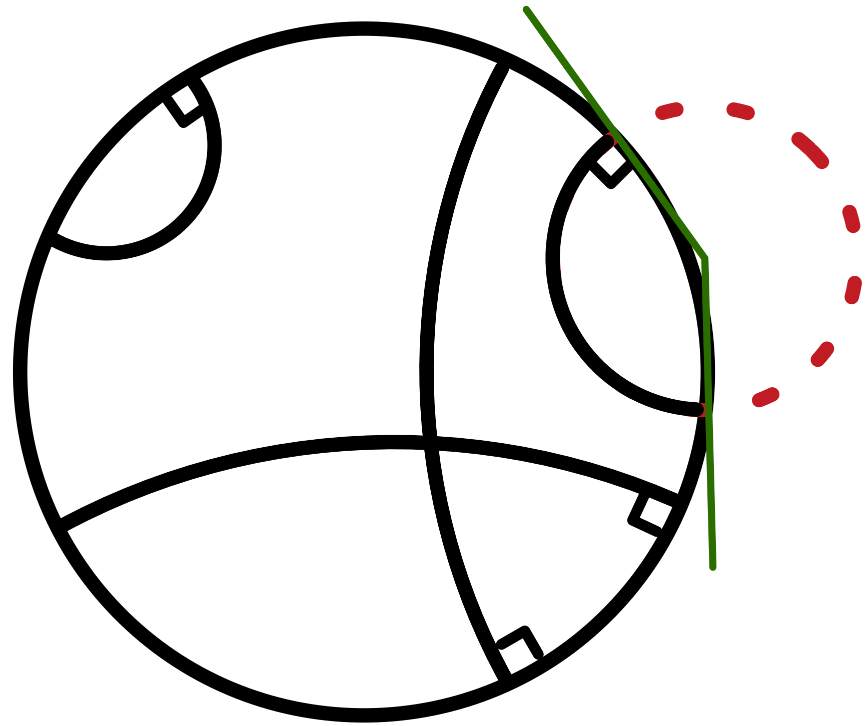


Straight lines in the Poincaré disk are represented by arcs of circles that meet the boundary circle at 90°.

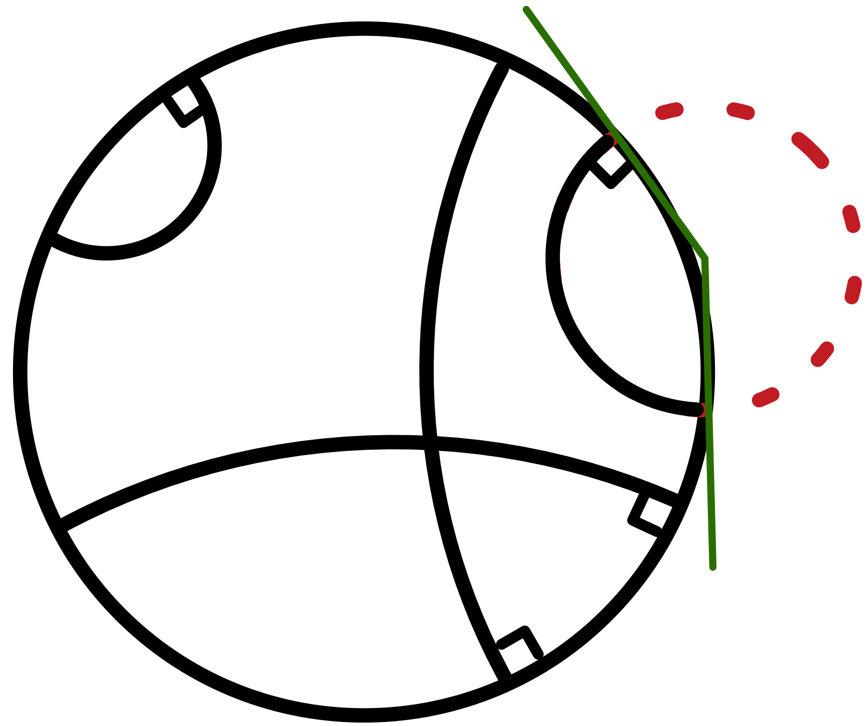


Problem:

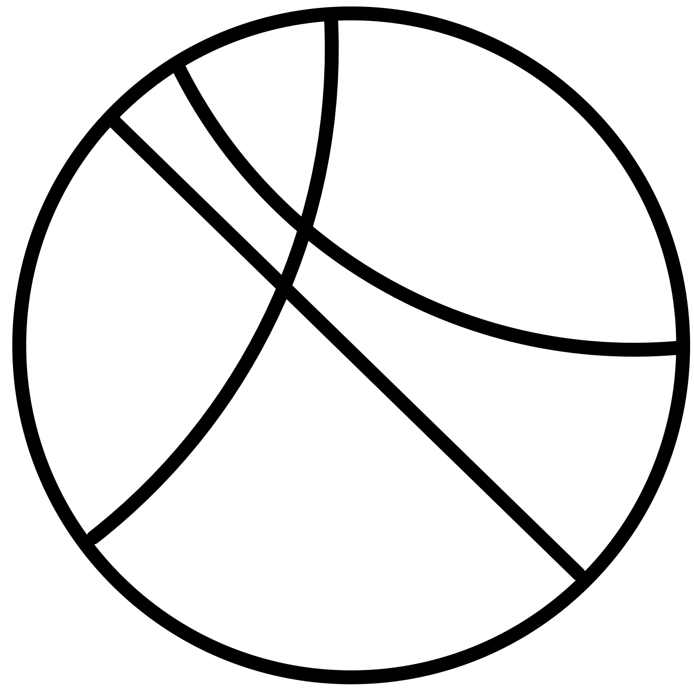
Sketch a few hyperbolic straight lines on a Poincaré disk.



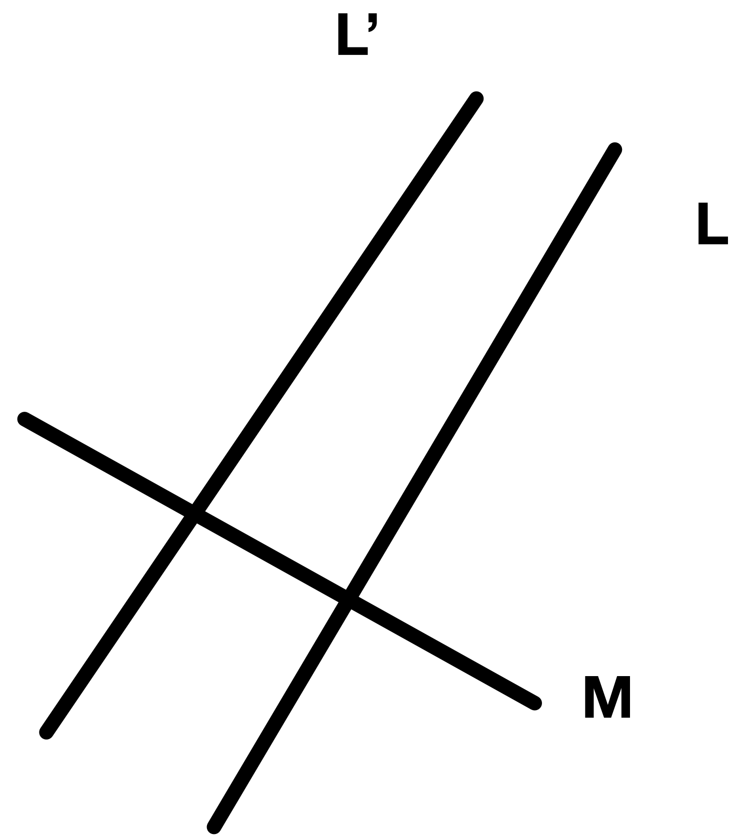
How do these lines act differently than those on the sphere or the Euclidean plane?



Notice: Lines that start near each other going in the same direction eventually move apart.

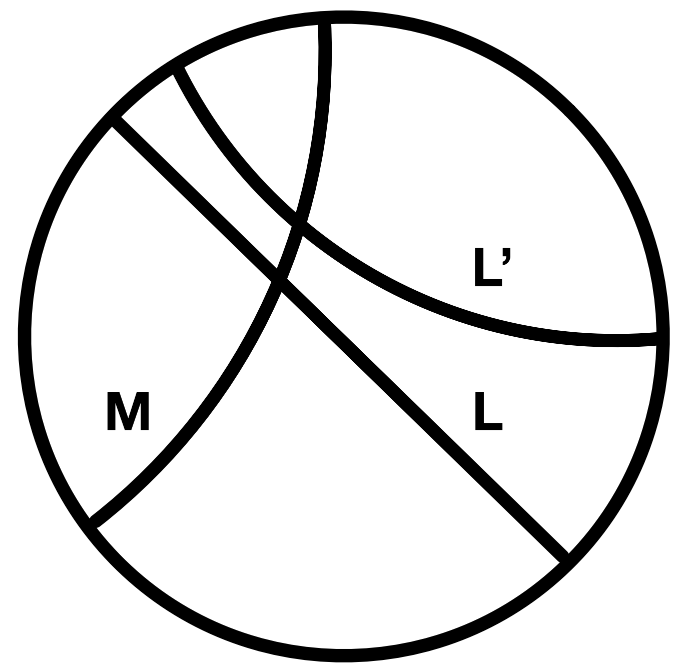


If hyperbolic lines L and L′ meet line M at different angles, do L and L′ have to meet?

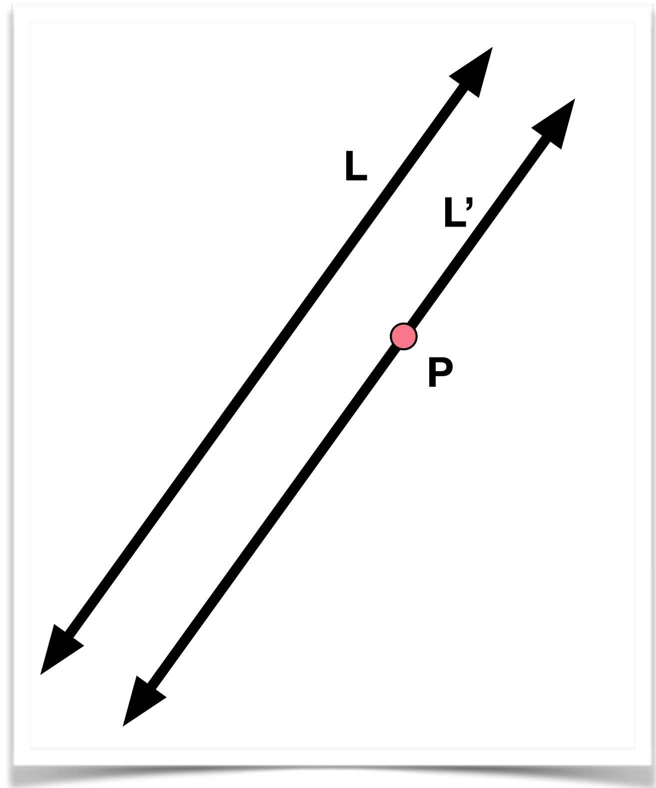


No

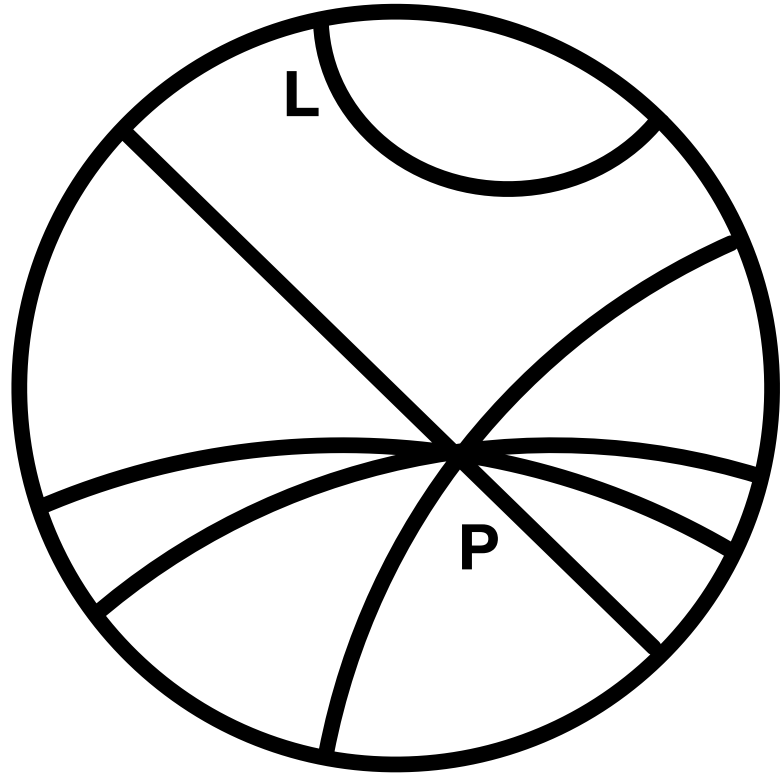
L and L′ do not have to meet.



In Euclidean space, given a point P and a line L, there is only one line through P that misses L.



In hyperbolic space, given a line L and a point P, there are an **infinite number** of lines going through P that miss L.

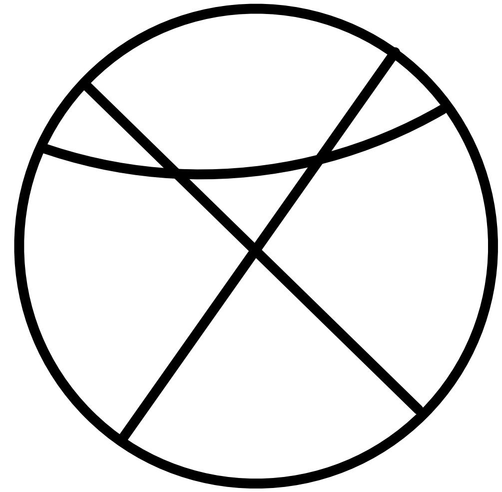


On the plane, the angle sum of a triangle is equal to 180°.

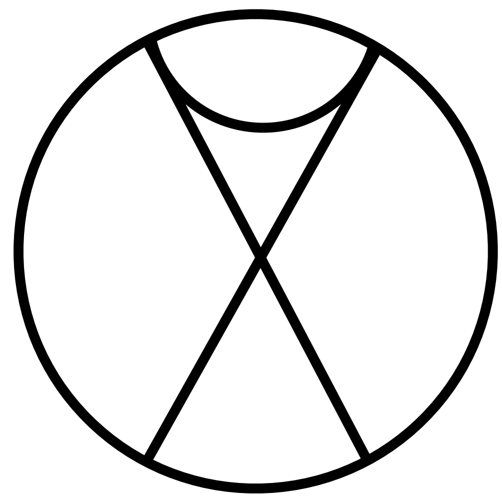
On the sphere, the angle sum of a triangle is greater than 180°

How do you think the angle sum behaves in hyperbolic space?

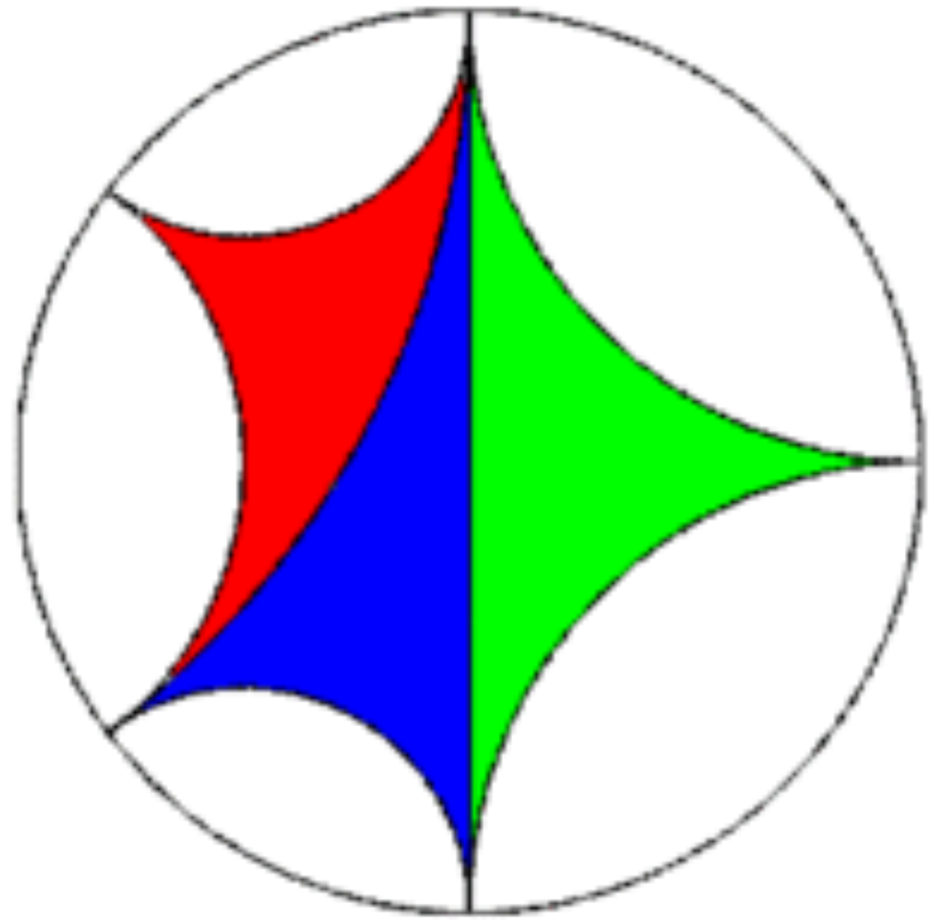
For small triangles, the angle sum is a bit less than 180°.



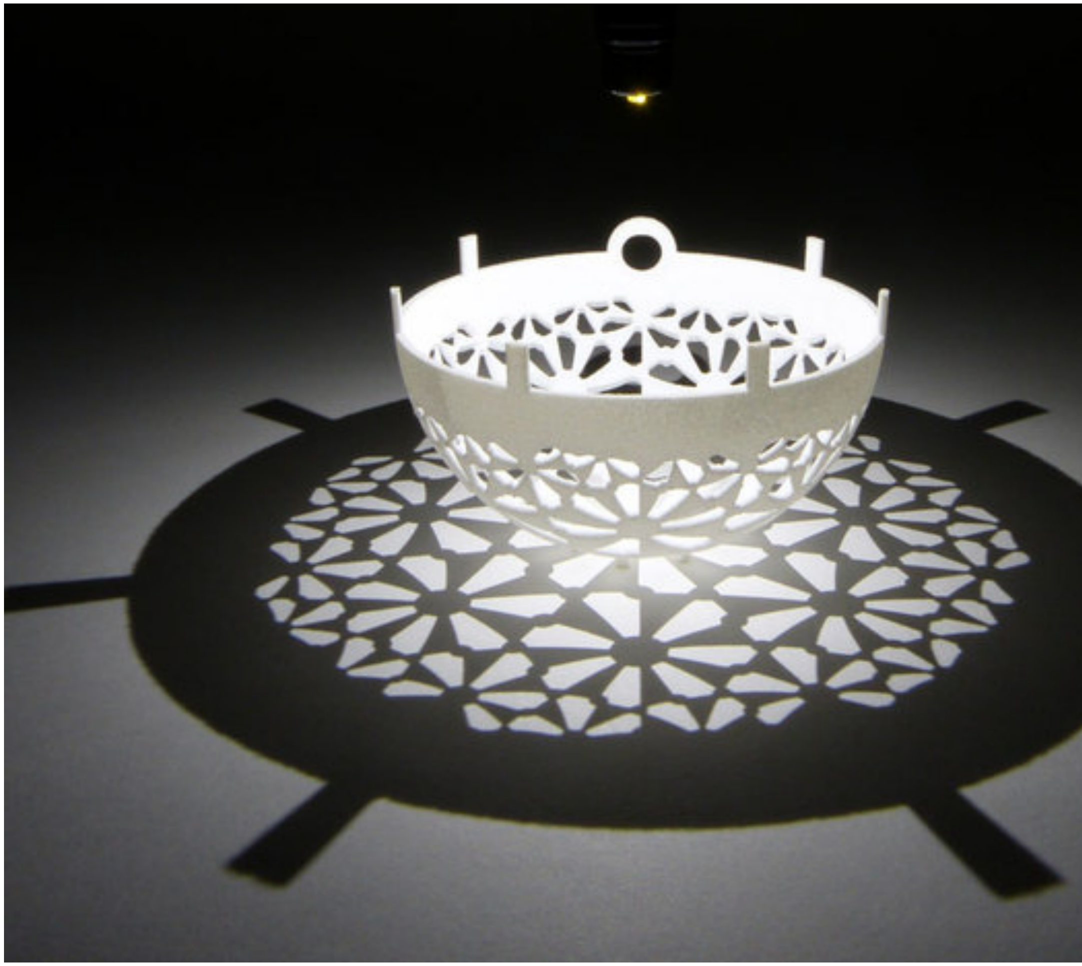
For large triangles, the angle sum can be much smaller.



If all three corners of the triangle are near infinity, then the angle can be very close to 0.



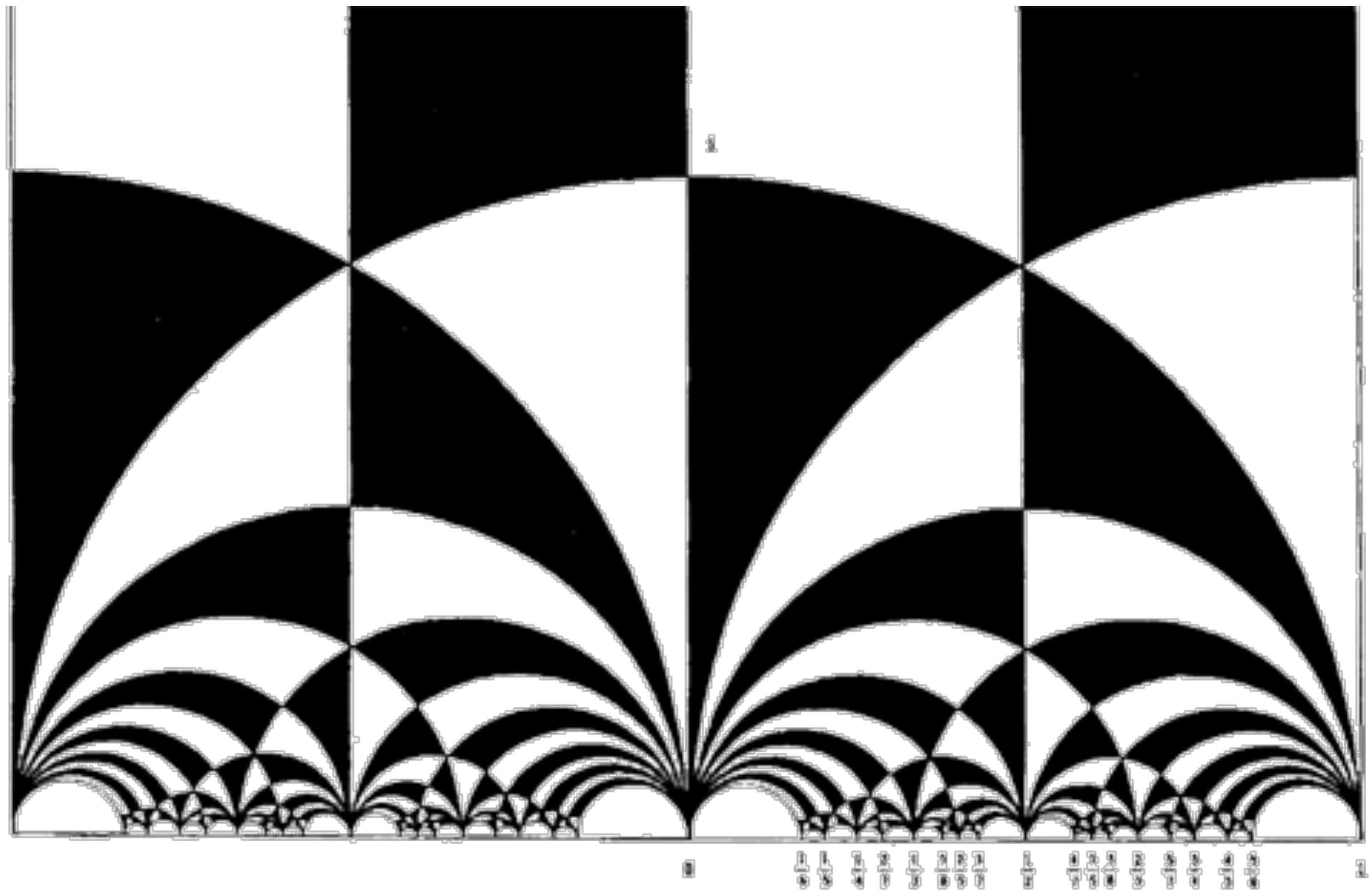
The shadow of this sculpture gives a picture in the Poincare disk.



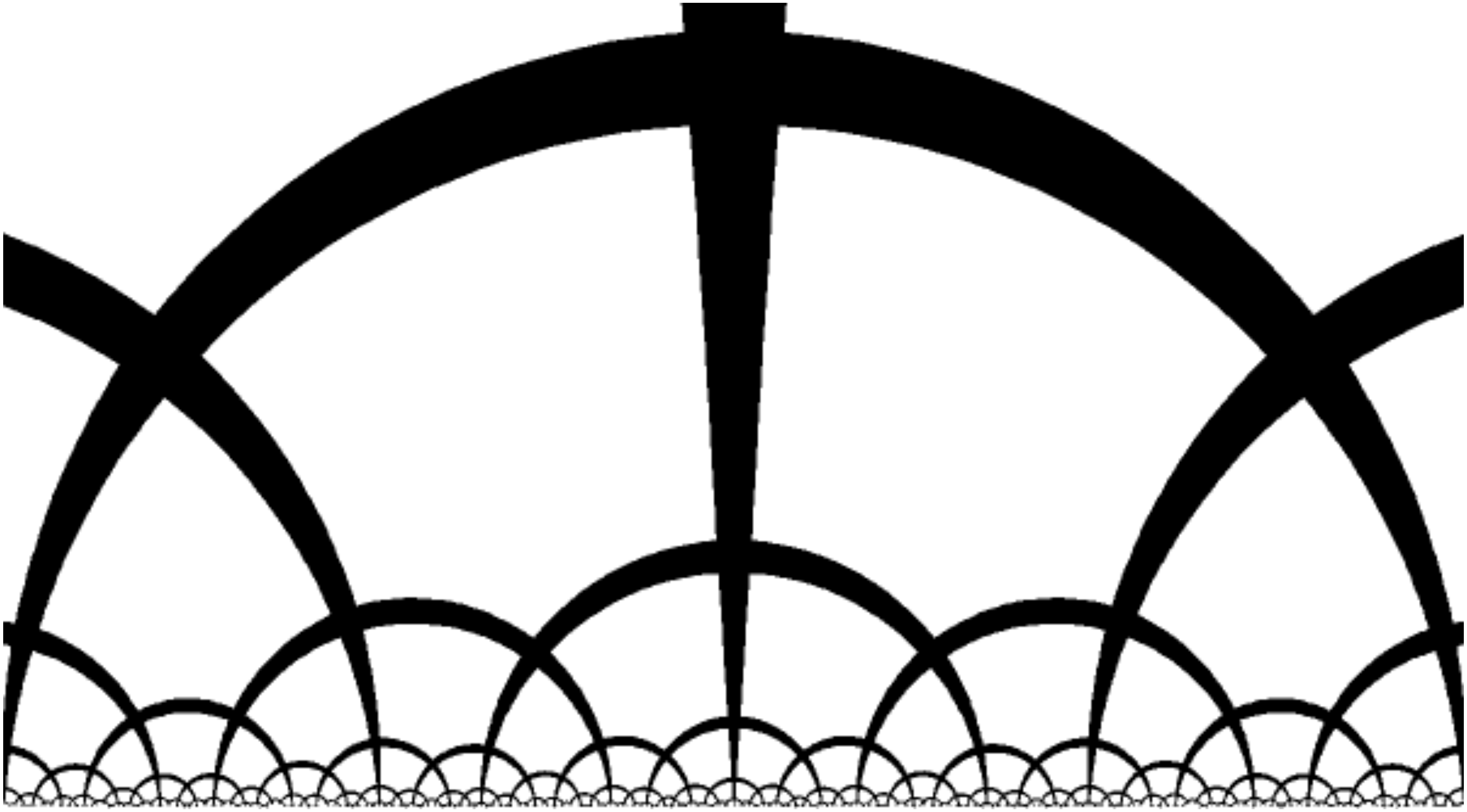
sculpture by Henry Segerman

sculpture by Henry Segerman

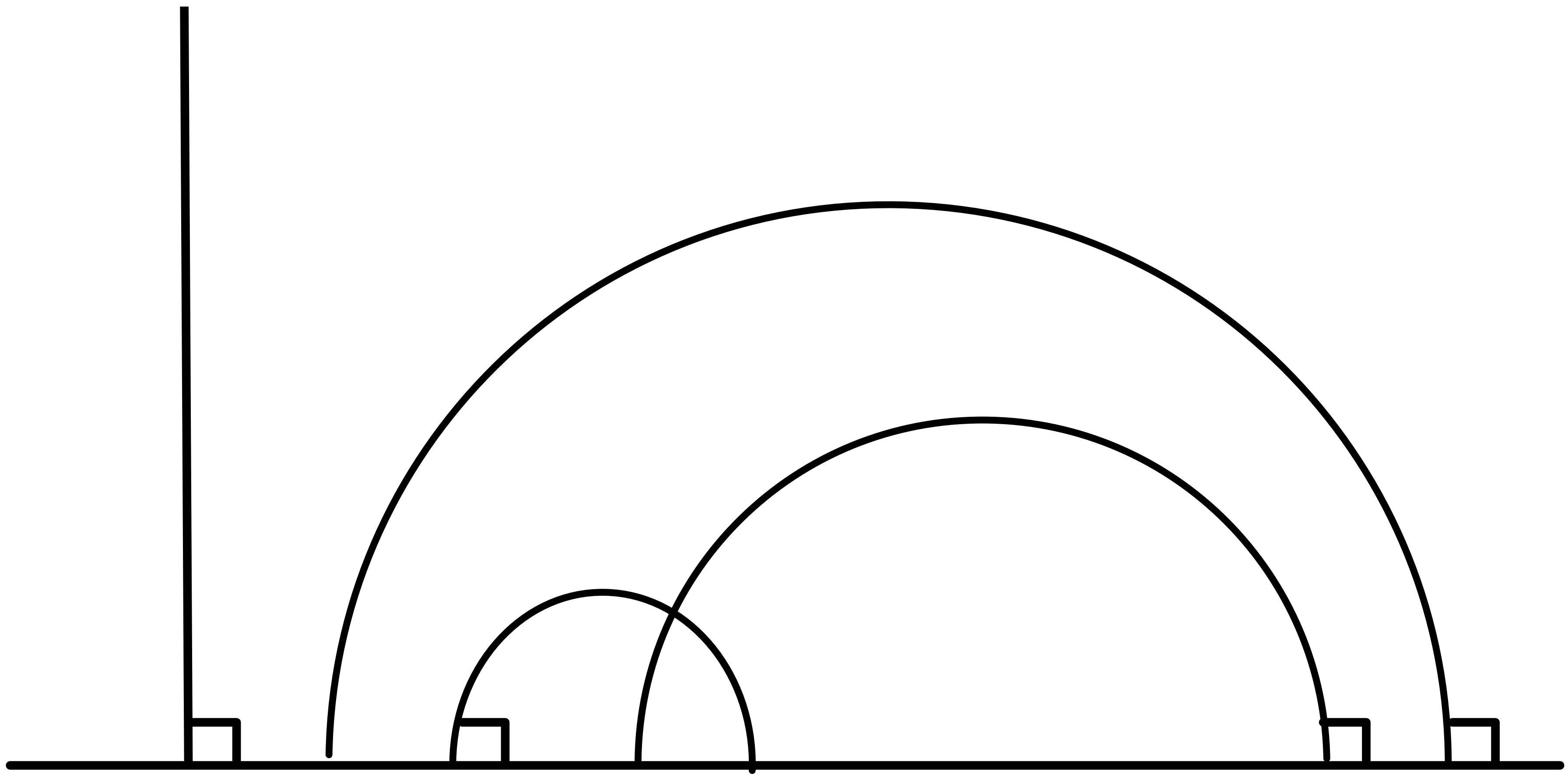
If we shine the light from a different angle, we get another “map” of hyperbolic space.



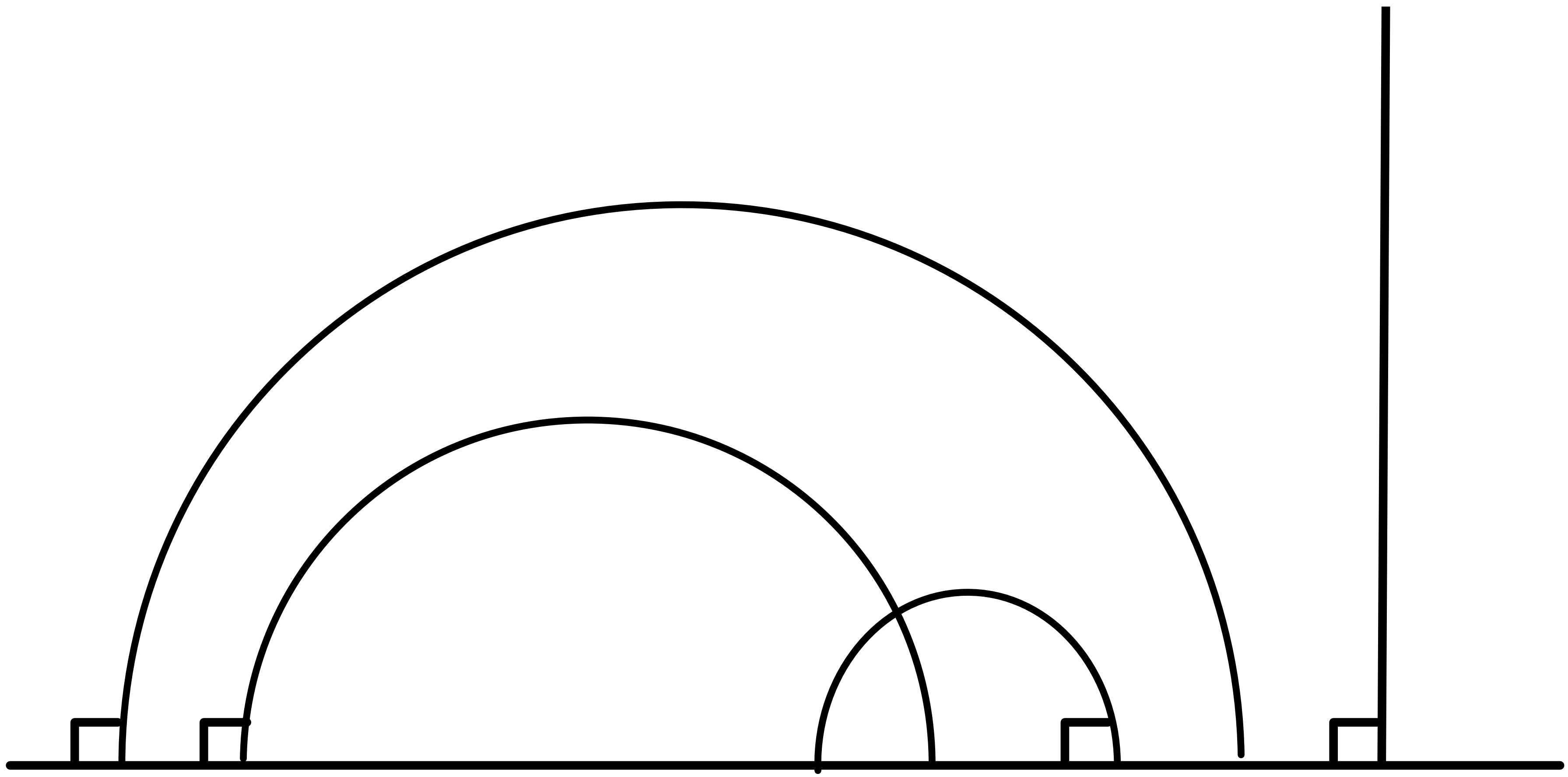
The Upper Half Plane model. The boundary at infinity is the x-axis.



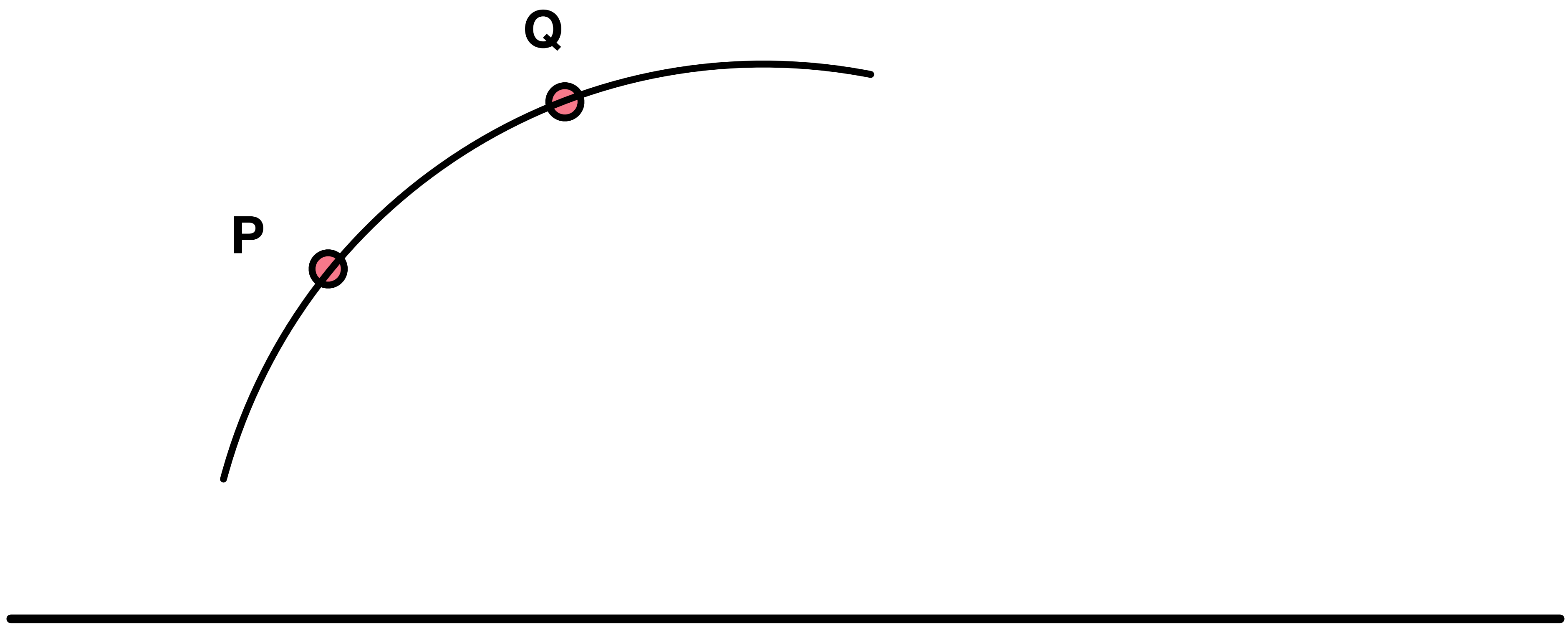
Distances are wrong, but angles are correct.



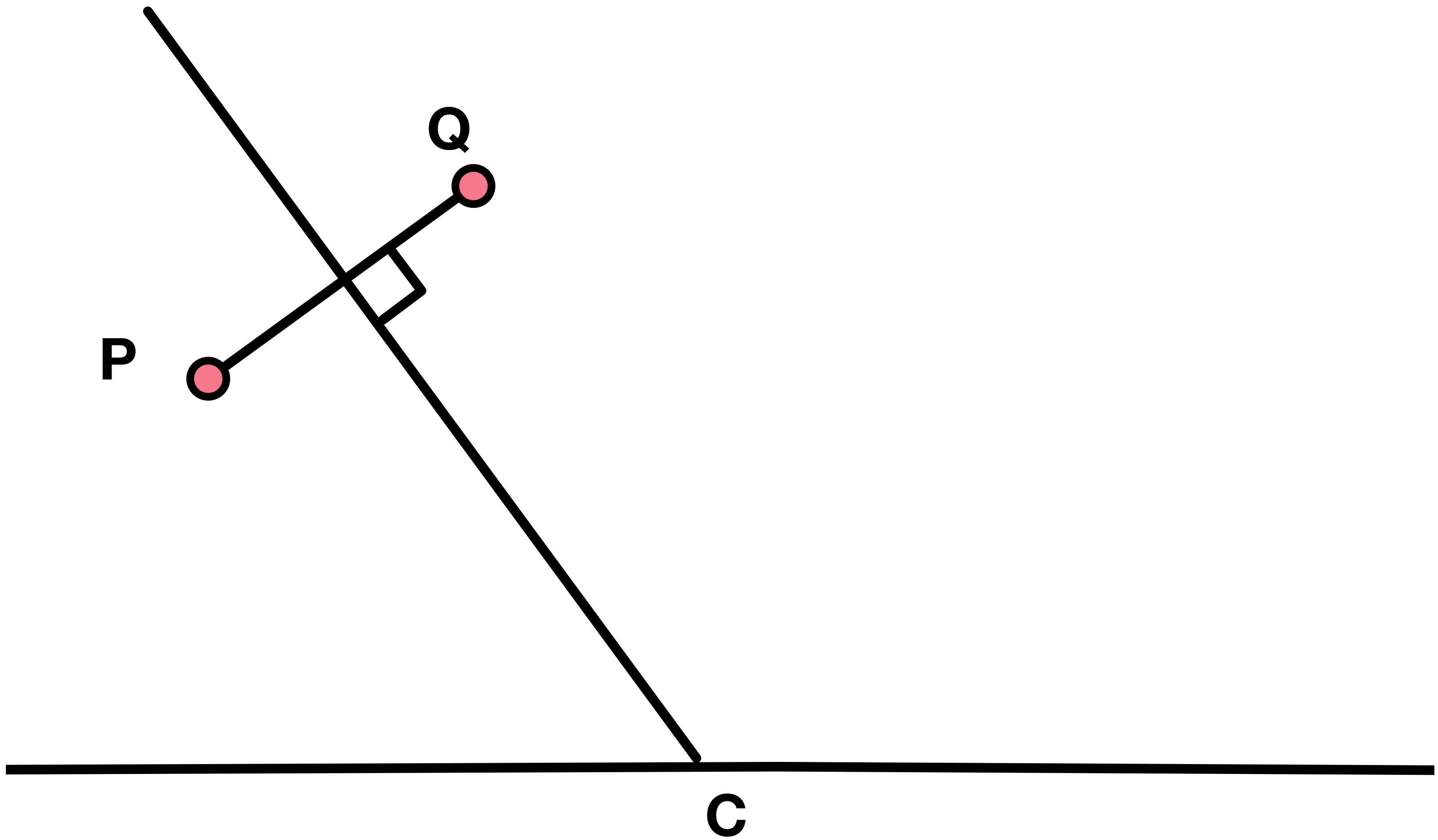
Lines are half circles meeting the boundary at 90°.



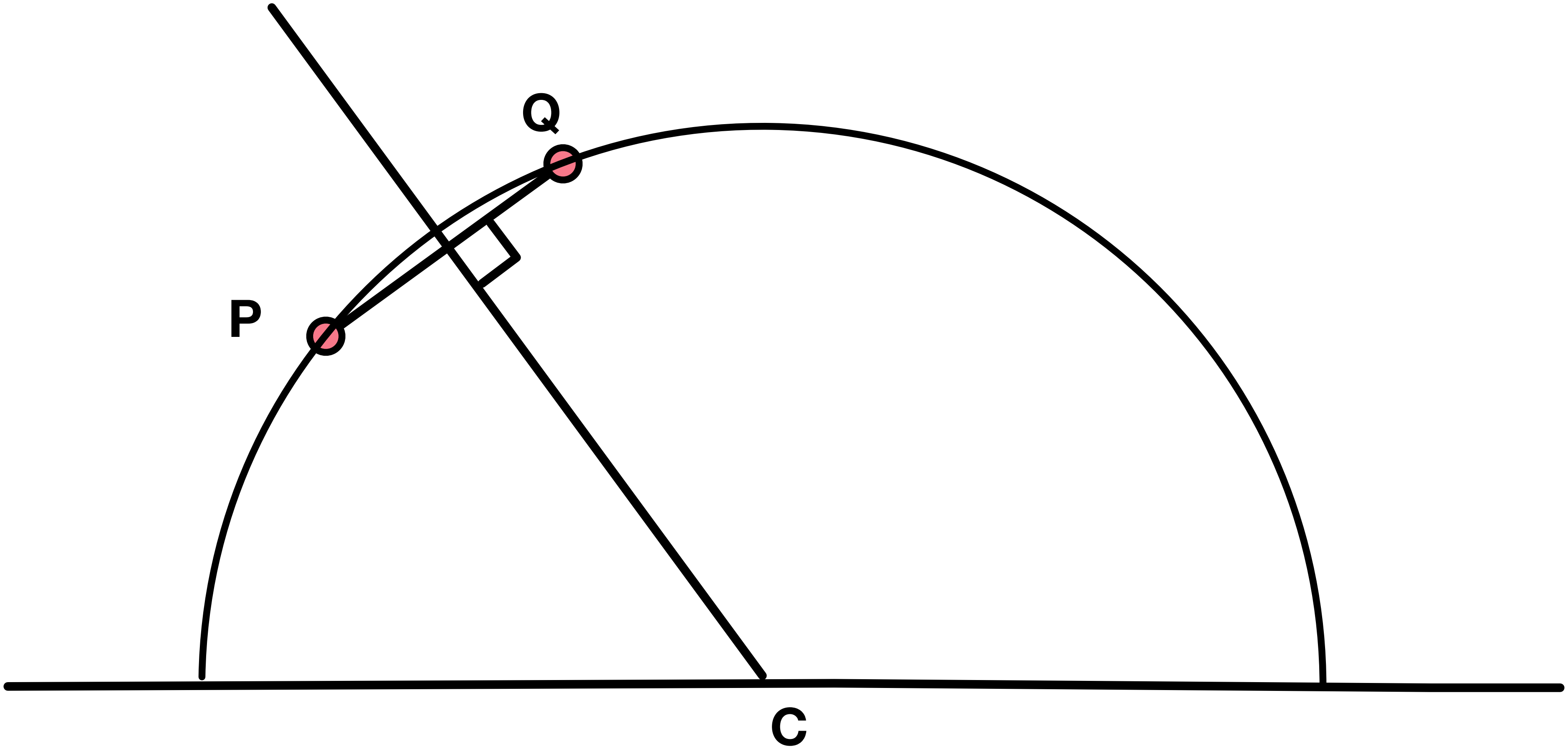
<u>Problem:</u> draw a few hyperbolic lines on the upper half plane



Given points P and Q, how could you use ruler and compass to find the hyperbolic line through them?

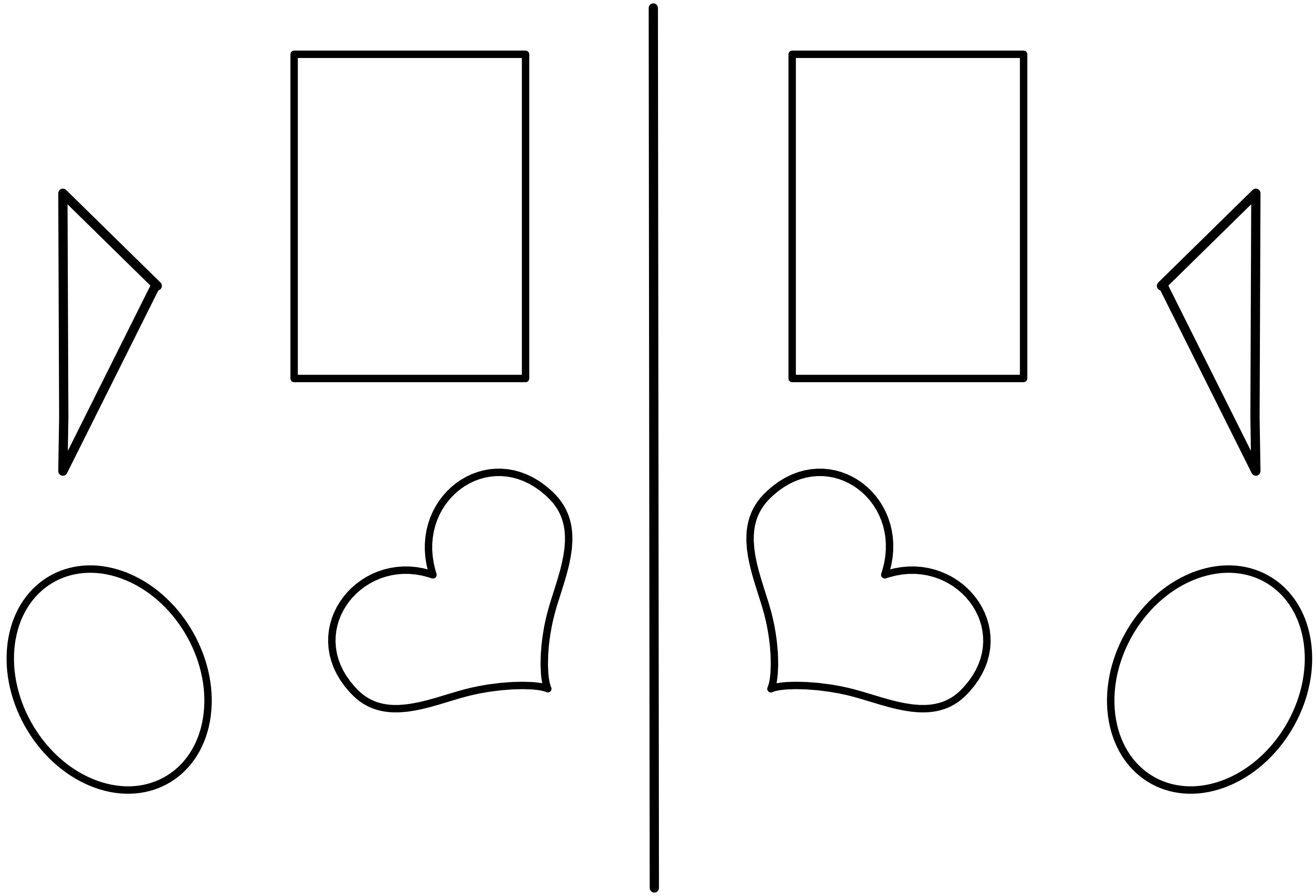


Draw the Euclidean line segment PQ and find its perpendicular bisector.

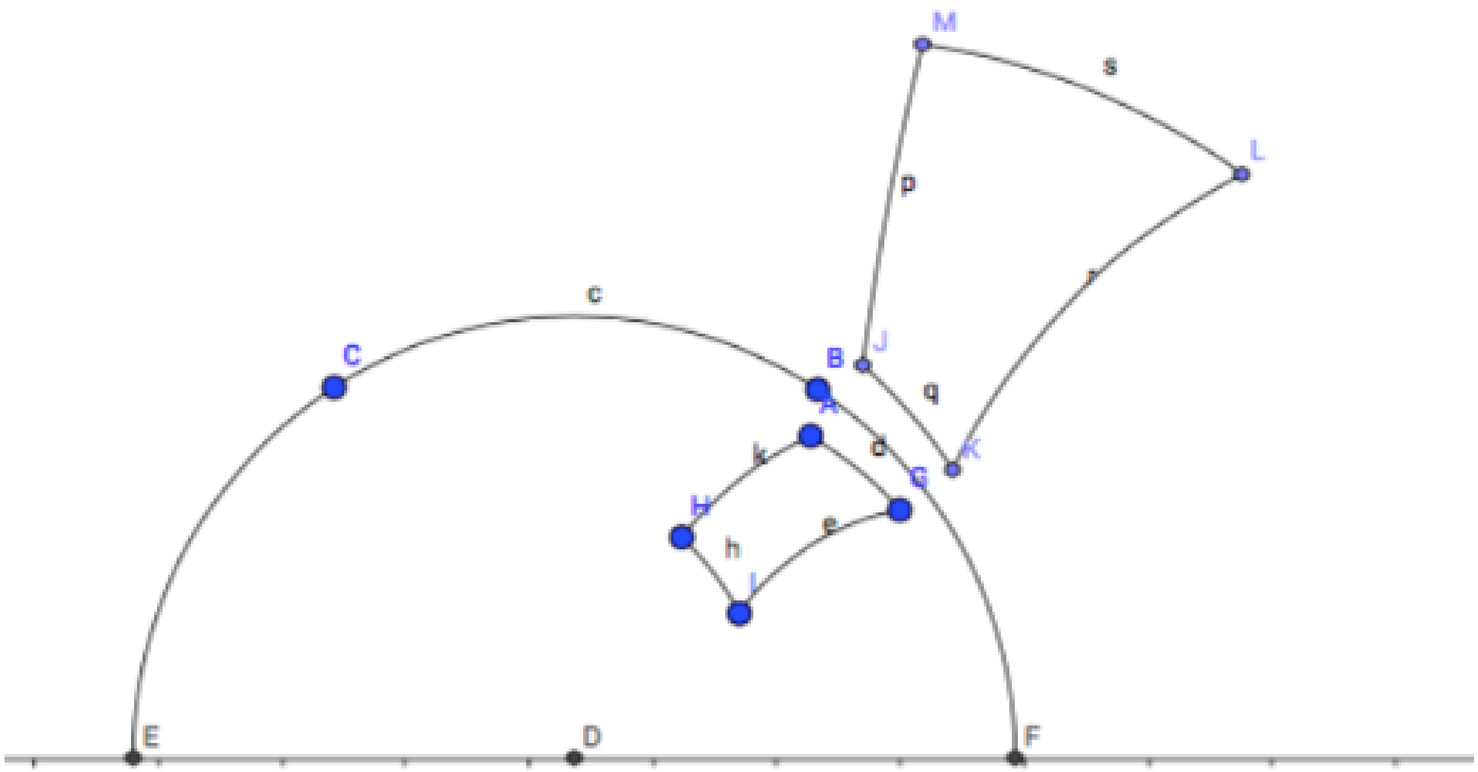


Here is the hyperbolic line though P and Q.

More facts about hyperbolic space



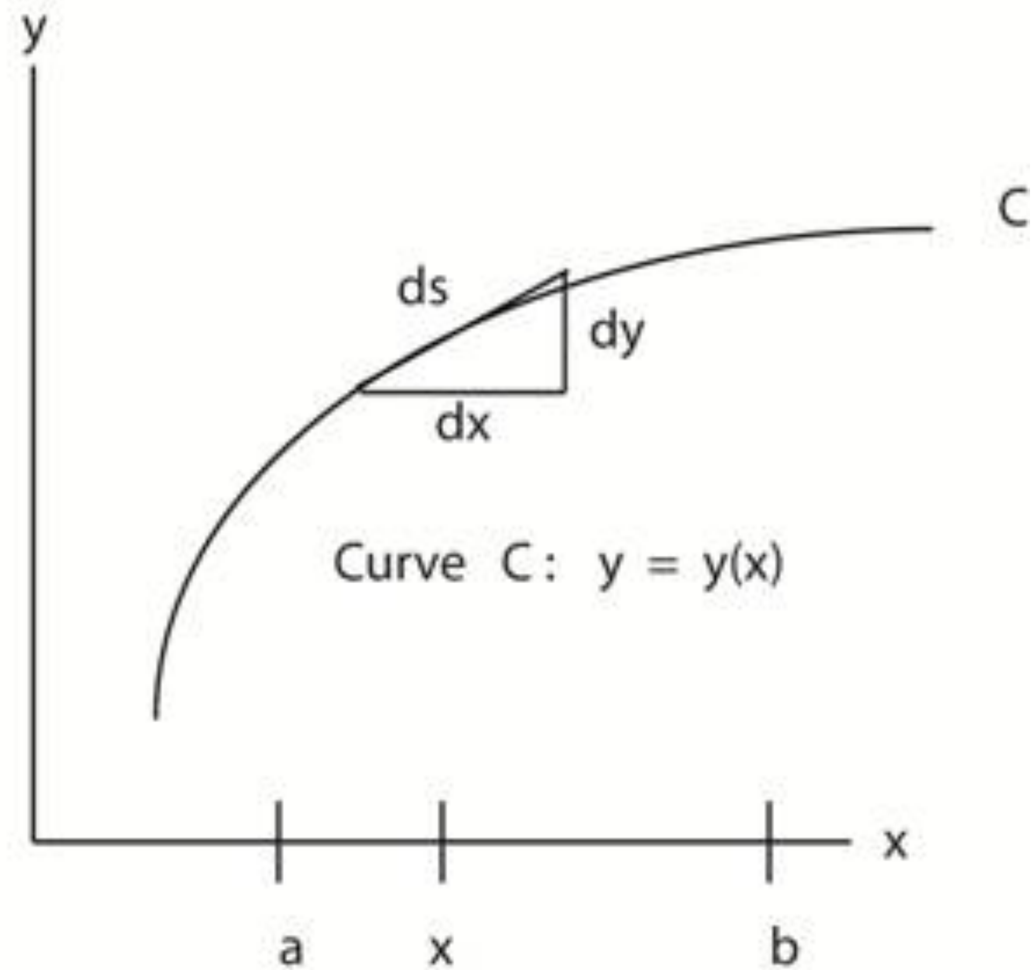
Reflections: In the Euclidean plane, we can reflect across a line.



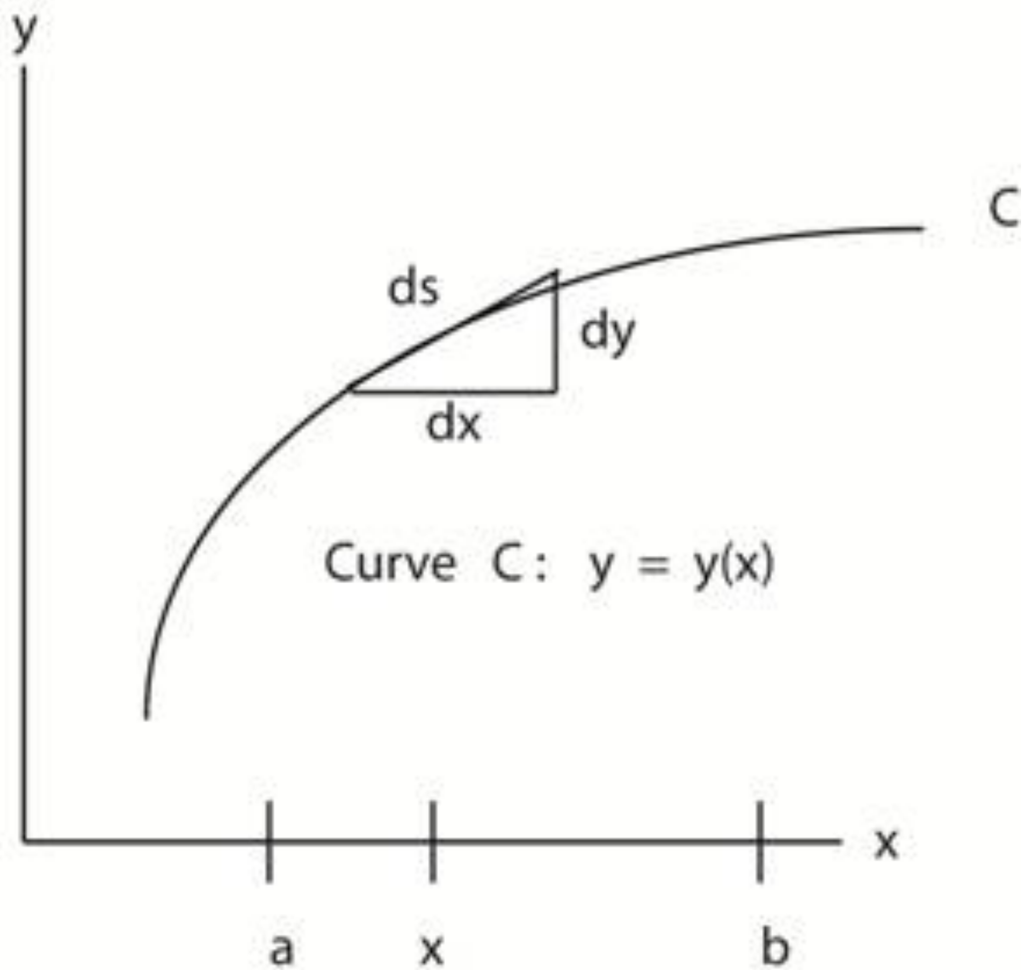
made with GeoGebra

page by Jordi Arnau

To reflect across a hyperbolic line, we use a circle inversion.

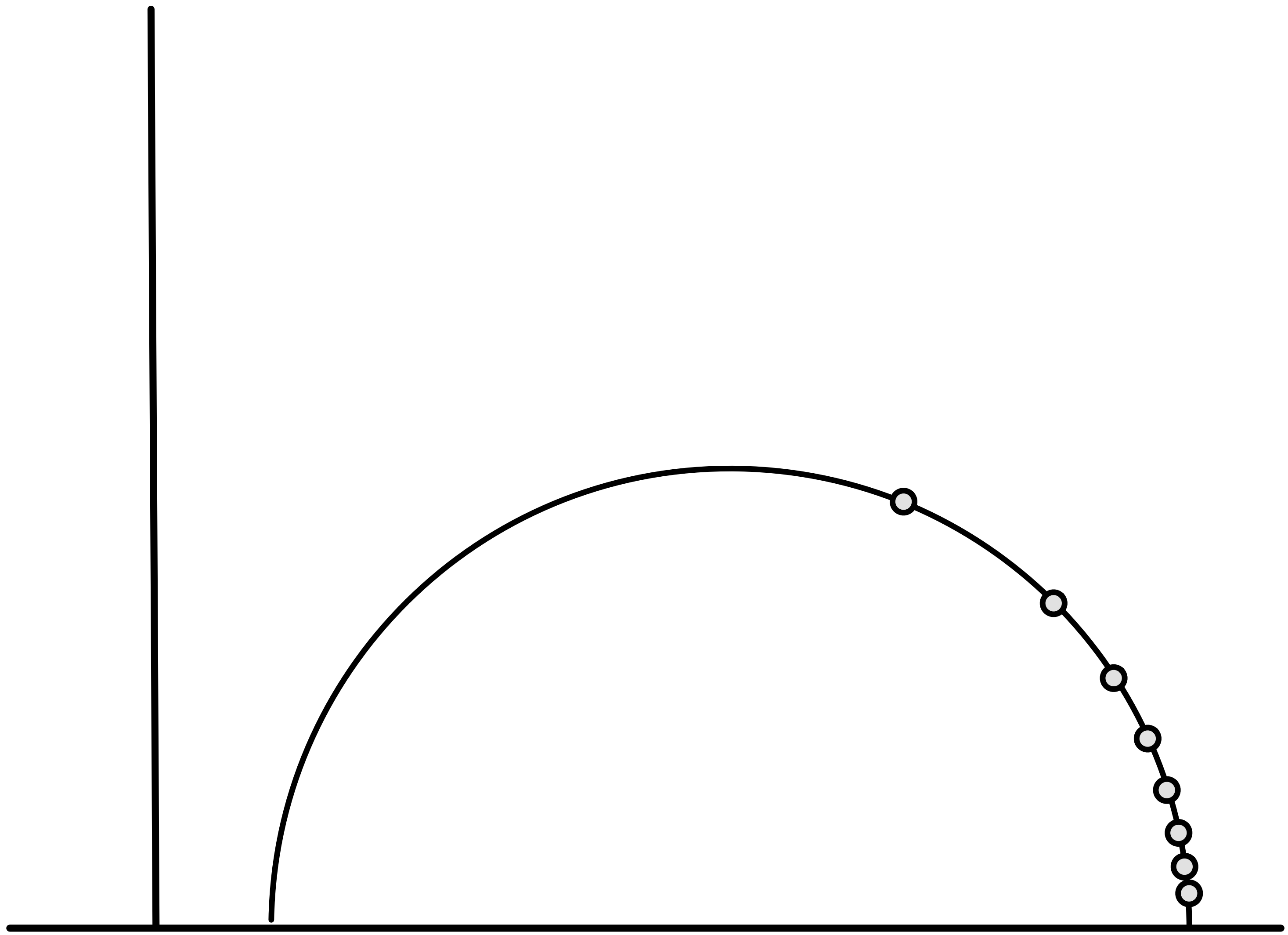


In the Euclidean plane, we measure distance by $ds^2 = dx^2 + dy^2$.

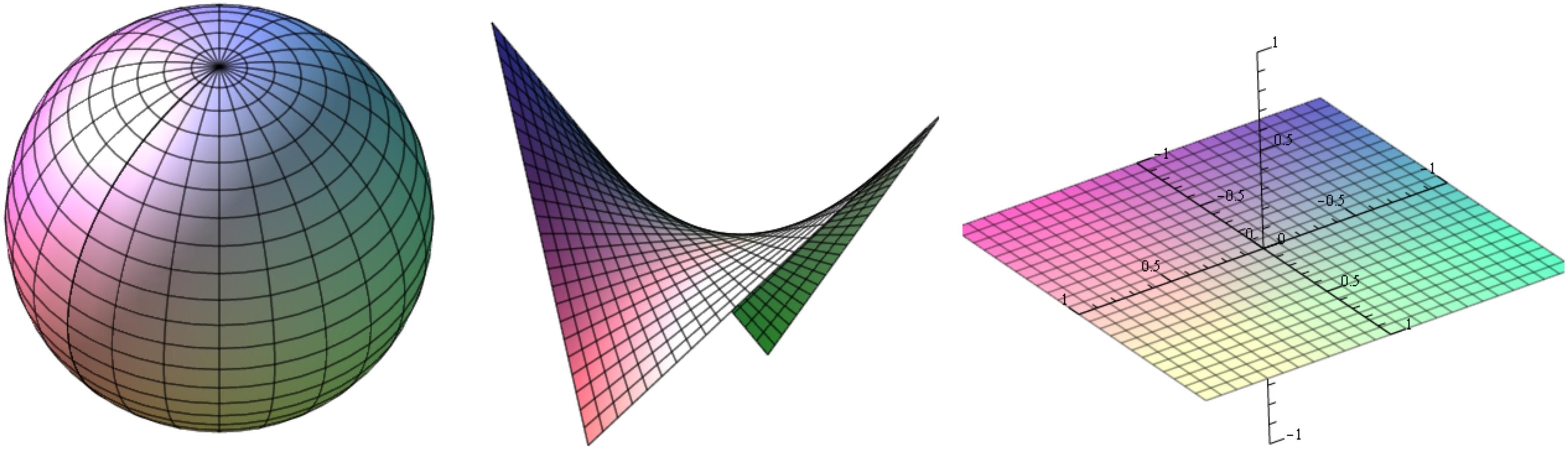


In the hyperbolic upper half plane,

we use $ds^2 = (dx^2 + dy^2)/y^2$.



As you get closer to the x-axis, the hyperbolic distance between points get larger.

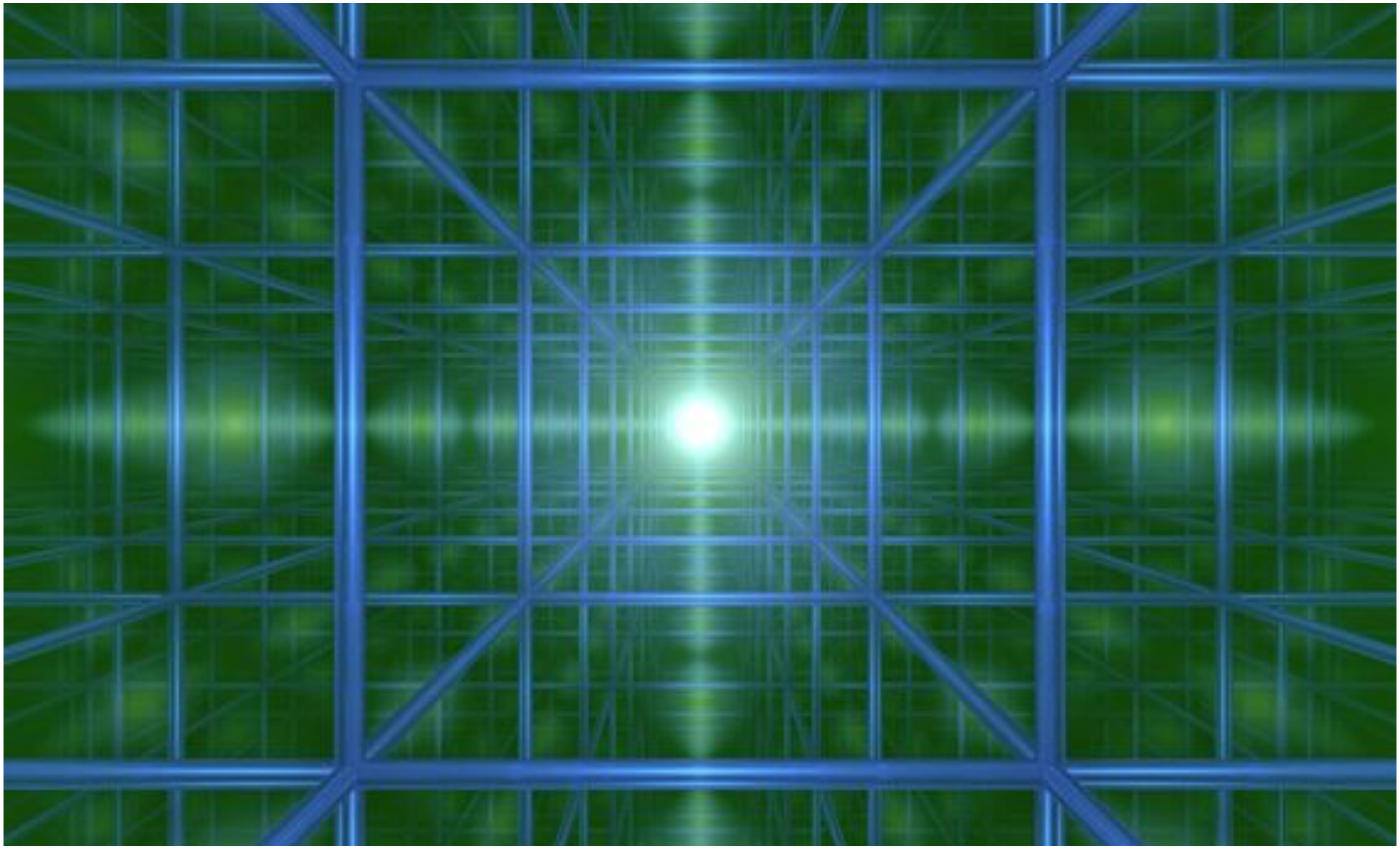


picture: Jonah Miller

Summary: three 2-dimensional geometries

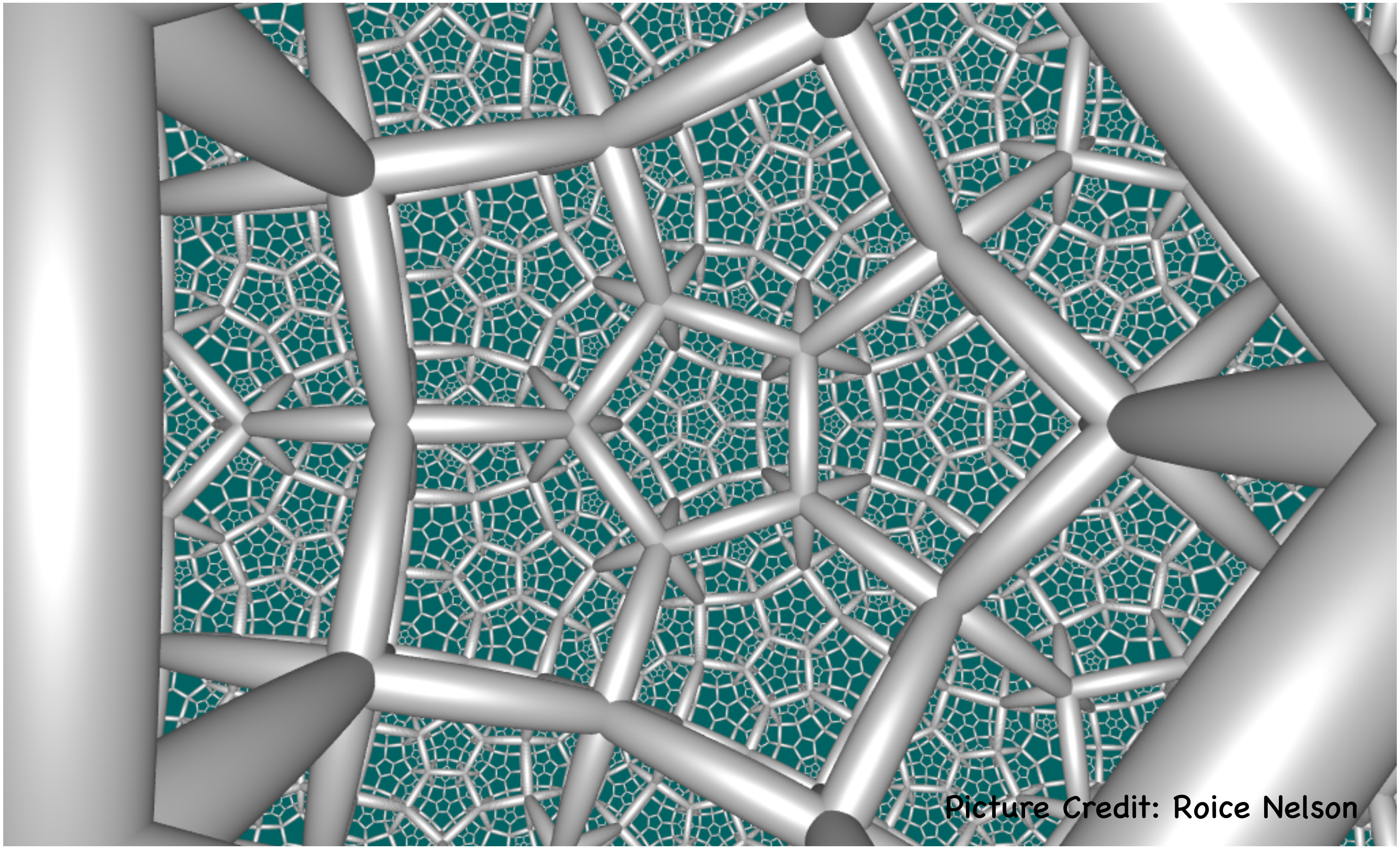
In 3 dimensions, there are actually 8 geometries.

Here are 2 of them:



Euclidean 3-space

Hyperbolic 3-space
(the upper half space model)



Hyperbolic 3-space

Some History

Ancient Greece

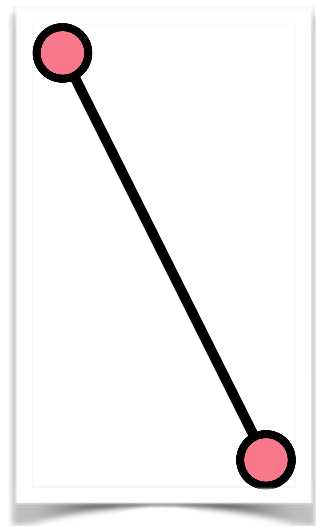
2400 years ago:

Euclid

5 Axioms to describe geometry

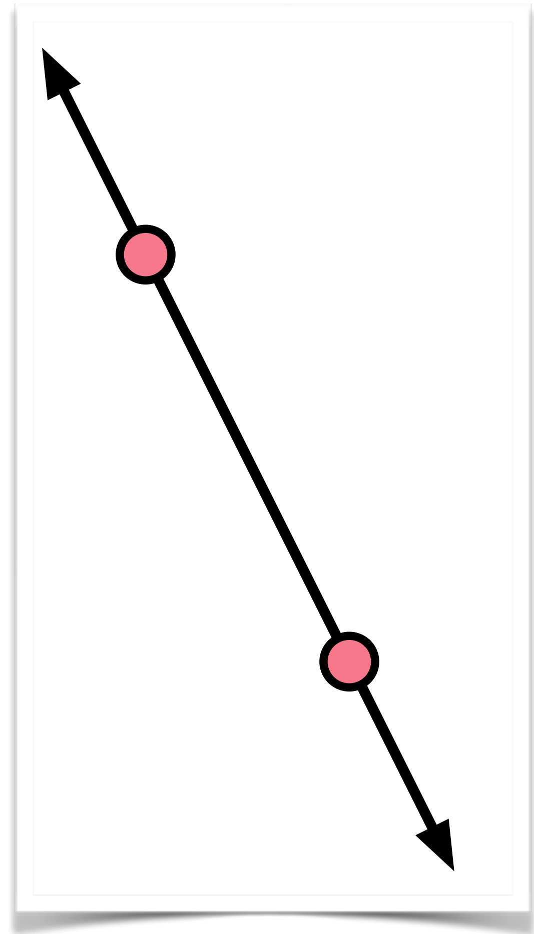
Euclid's 1st axiom

A straight line segment can be drawn through any two points.



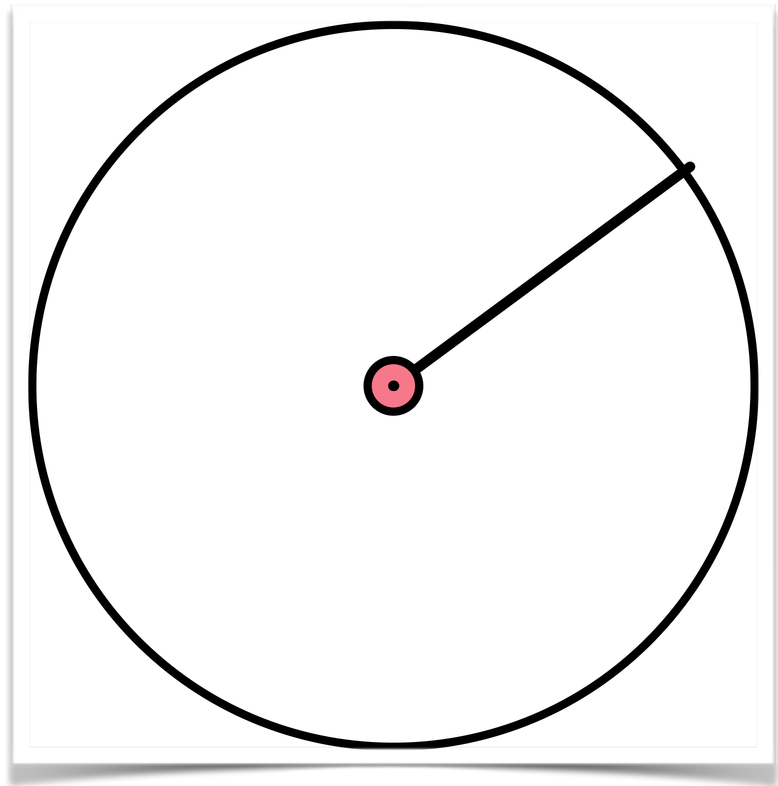
Euclid's 2nd:

Any straight line segment can be extended indefinitely to a straight line.



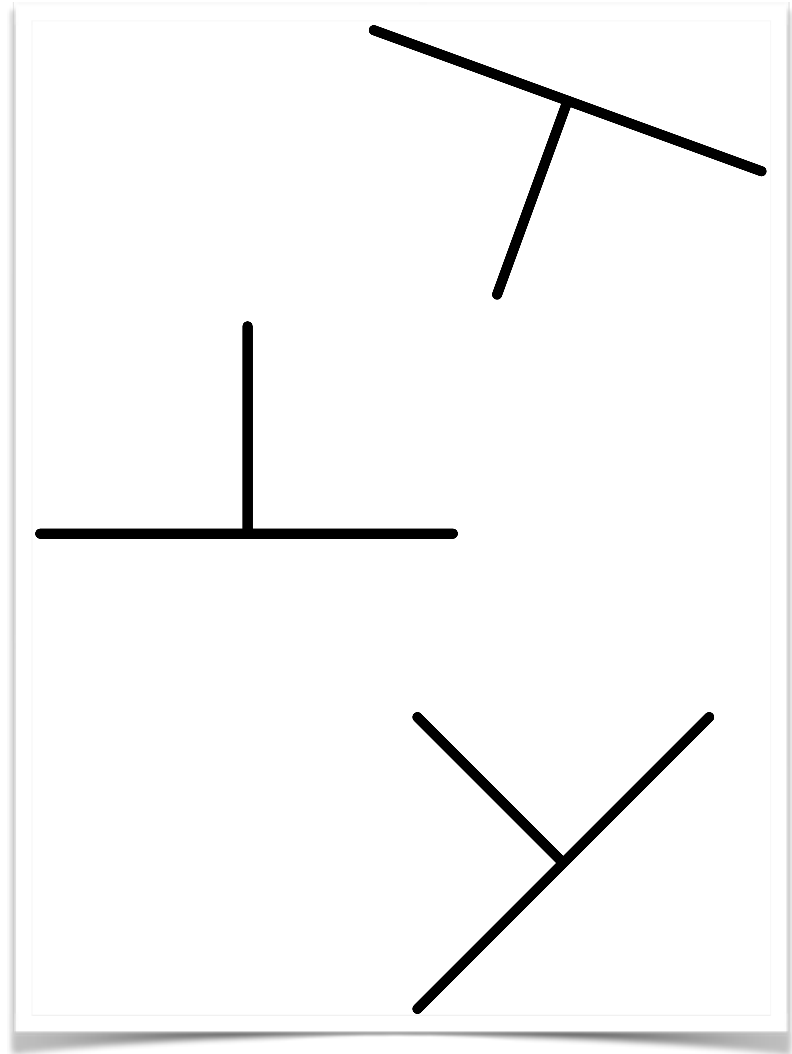
Euclid's 3rd:

A circle may be drawn with any point as center and any radius.



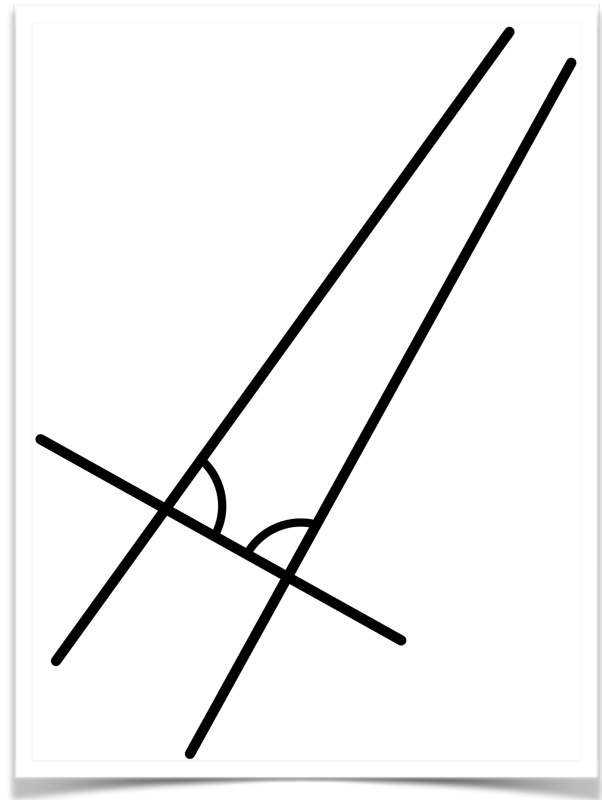
Euclid's 4th:

All right angles are equal.
(A right angle is the angle at the foot of a perpendicular.)



Euclid's 5th:

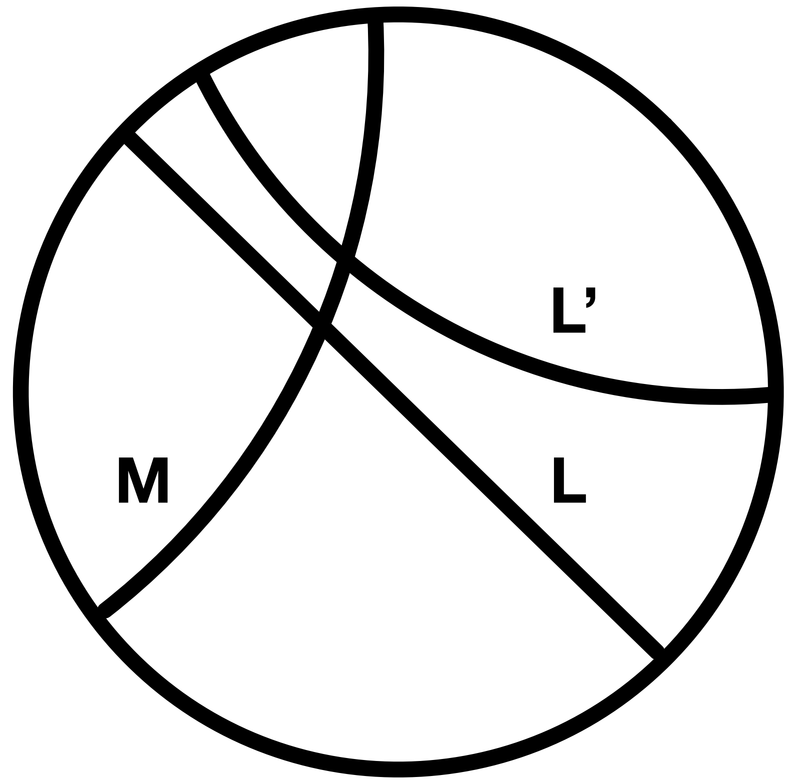
If lines L and L′ meet line M so that the interior angles do not add up to 180°, then L and L′ must intersect.



For over two thousand years, mathematicians wondered if the 5th axiom followed from the other 4.

But it doesn’t.

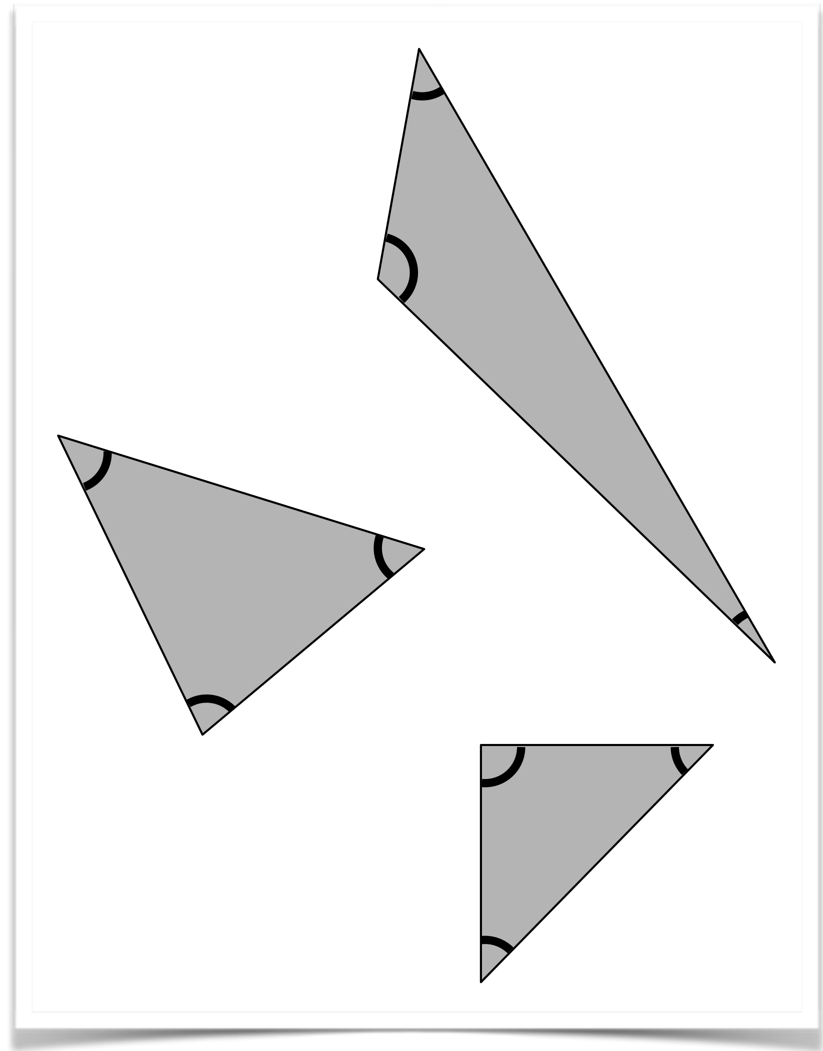
Axioms 1-4 work in the hyperbolic plane, but axiom 5 does not.



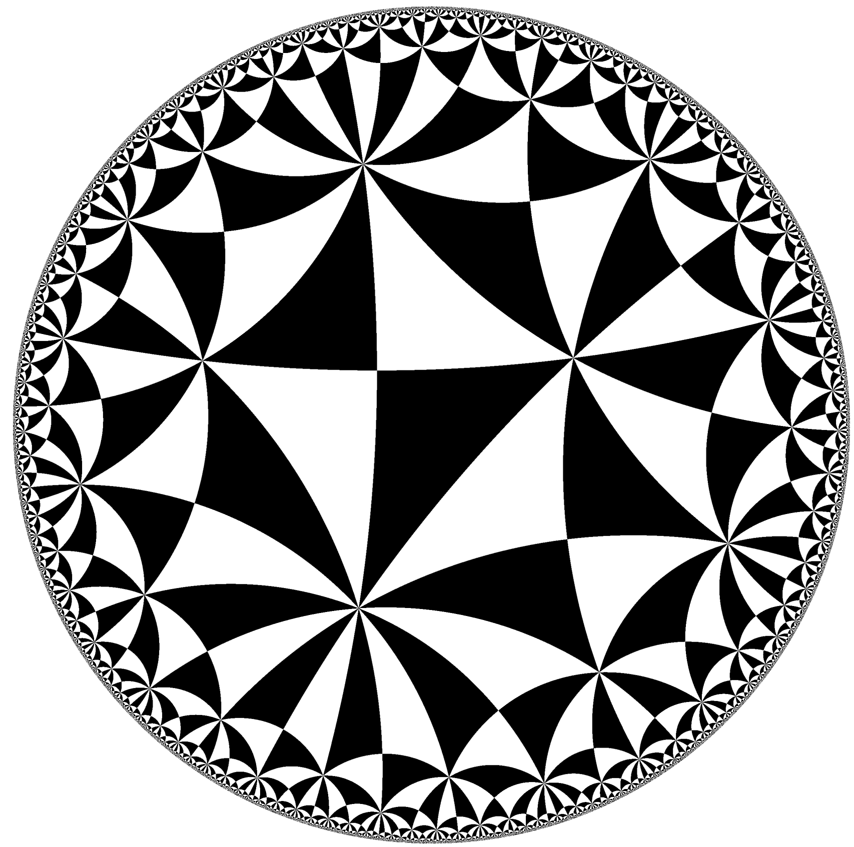
They found new axioms that were equivalent to Euclid's 5th:

Equivalent to Euclid's 5th:

The angle sum of any triangle is 180°.



In hyperbolic space, the angle sum of any triangle is less than 180°.

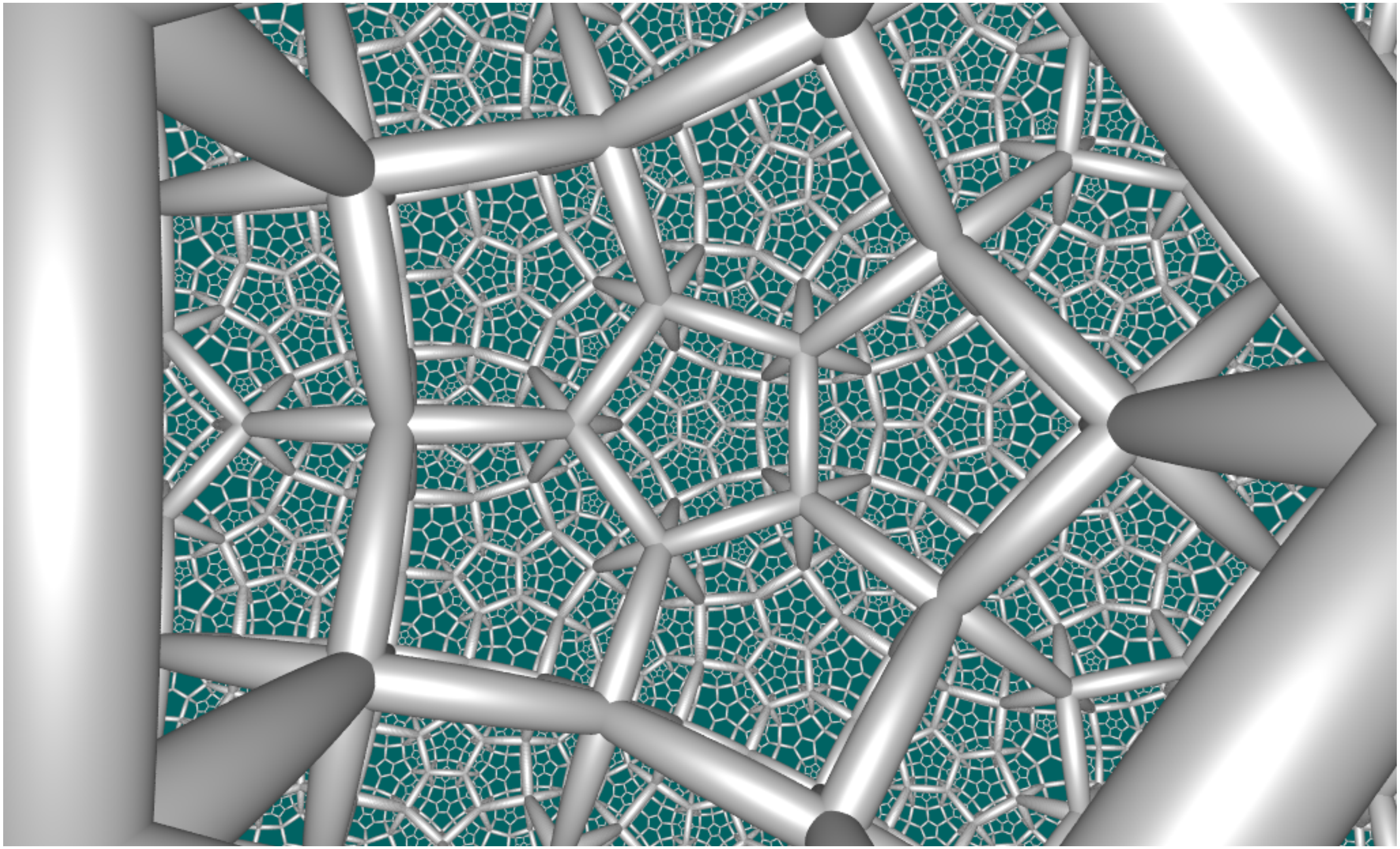


Some mathematicians who worked with Euclid's Elements:

- first known translation into Arabic: al-Hajjaj ibn Yusuf ibn Matar, 1300 years ago (الحجاج بن يوسف)
- al-Nairizi, al-Wafa' Buzjani, Al Kindi, Thabit Qurra, 1200 years ago (النيريزي , ابوالوفا بوزجانى , الكندي , ثابت بن قرة)
- al Haytham, al Din Tusi, al Khayyam, 1000 to 1100 years ago (الهيثم , الدين طوسى , الخيام)
- Girolamo Saccheri, Johann Lambert, Ferdinand Schweikart, Carl Gauss, Farkas and Janos Boylai, Nikolai Lobachevsky, 300-400 years ago

What about the sphere?

Euclid's axioms can be interpreted to work on the sphere, but then his proofs do not work. (Euclid actually missed a few details, like needing a unique line through two points.)



شكرا جزيلا !

Some cool links:

1. https://www.youtube.com/watch?v=eGEQ_UuQtYs

2. http://cs.unm.edu/~joel/NonEuclid/

3. http://www.geogebra.org/m/1477903

4. https://www.youtube.com/watch?v=xVE18hh4xDw&nohtml5=False

5. https://www.youtube.com/watch?v=AKotMPGFJYk

6. https://www.youtube.com/watch?v=Hwi_FGkgloo