SARSI 2016

First Week Lectures

Math – Kim Whittlesey

Lecture 2

Tilings in non-Euclidean geometry

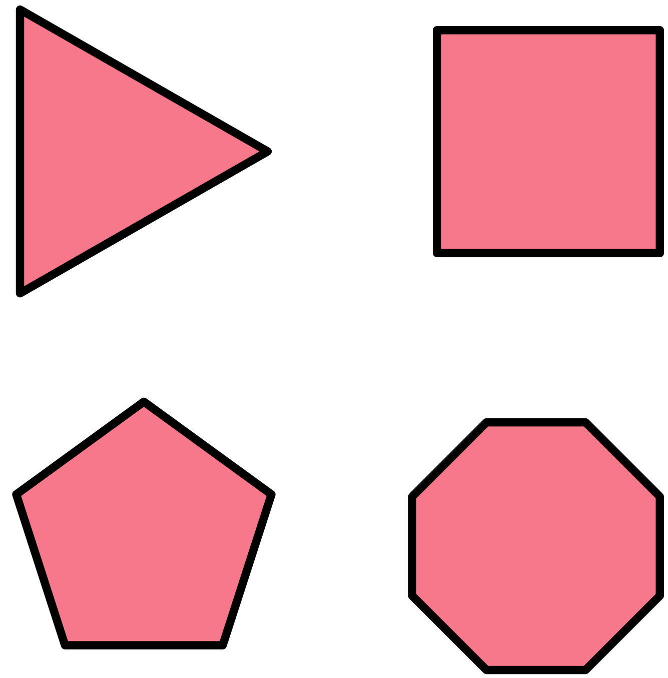
الزليج

Tilings of the Euclidean plane

Regular Tilings

Regular polygon:

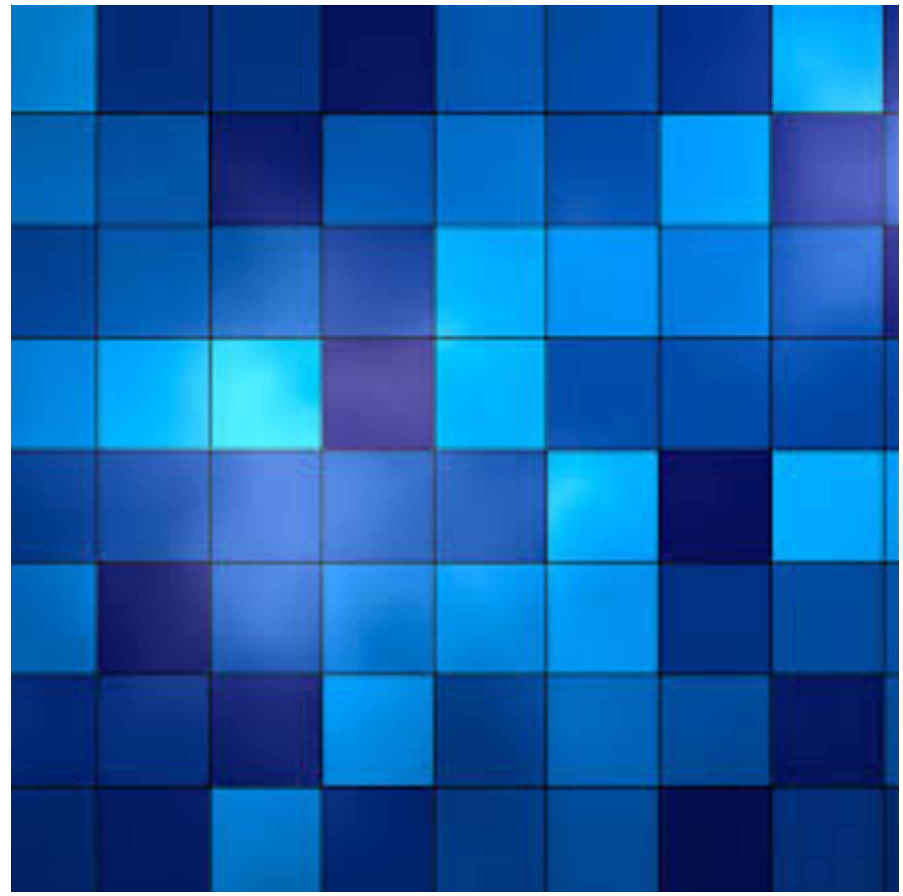
all sides have the same length, all angles have the same size.



Regular Tilings:

Only use one kind of regular polygon.

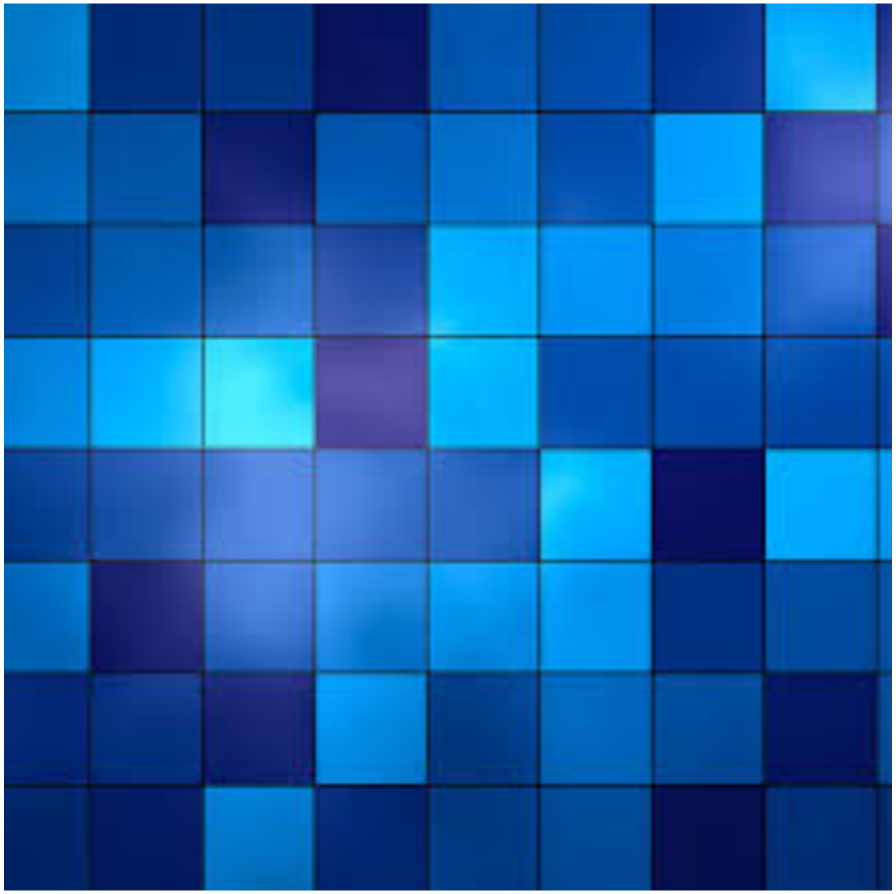
Polygons meet edge-to-edge.



This is not a regular tiling.

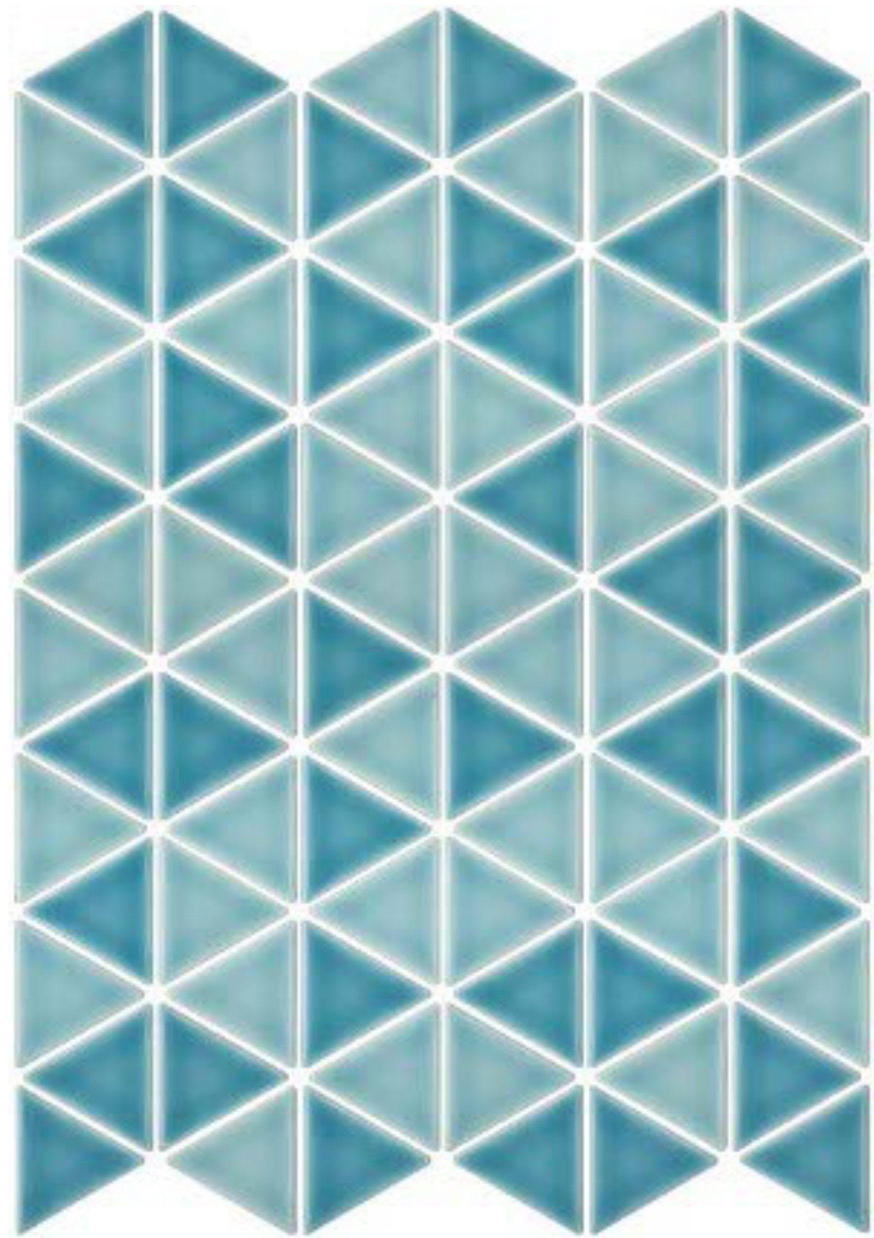
What are the three regular tilings of the Euclidean plane?

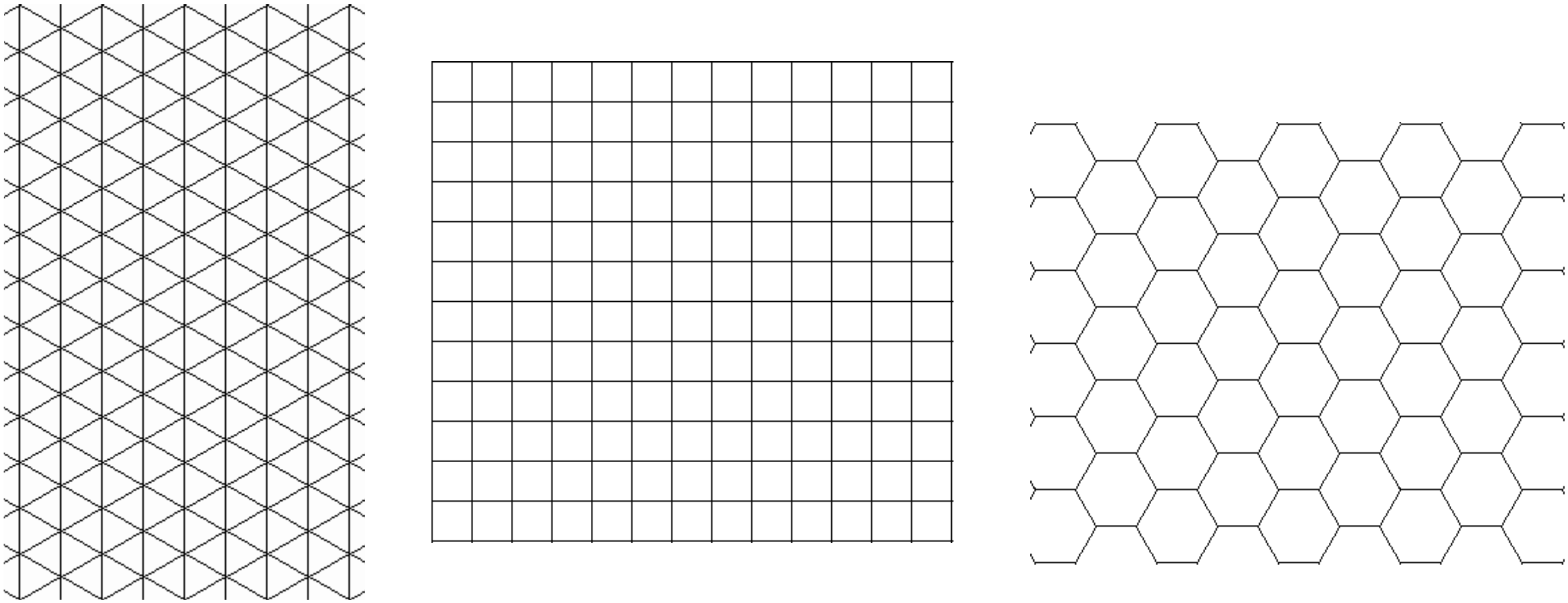
Tiling by squares



Tiling by regular hexagons

Tiling by regular triangles



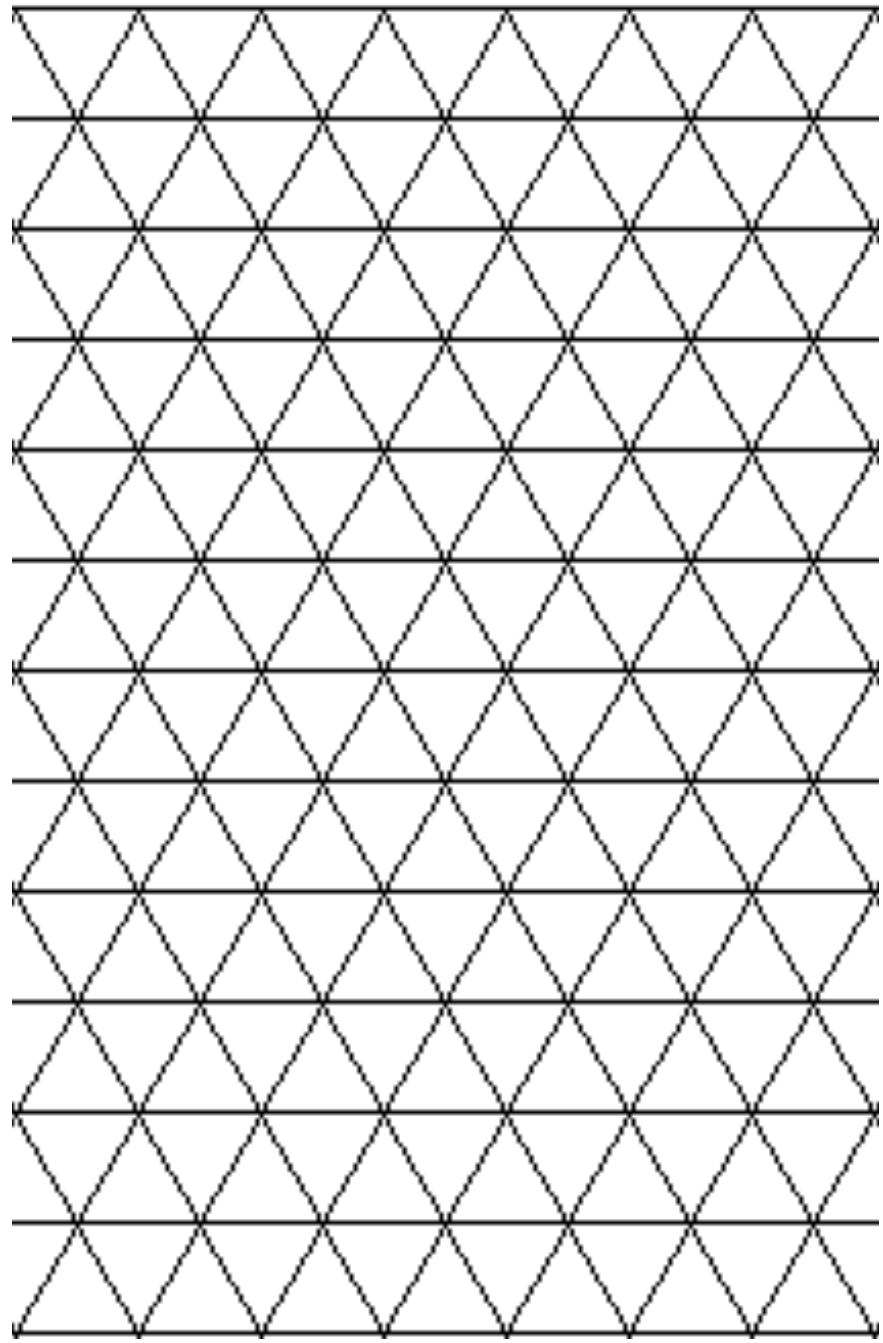


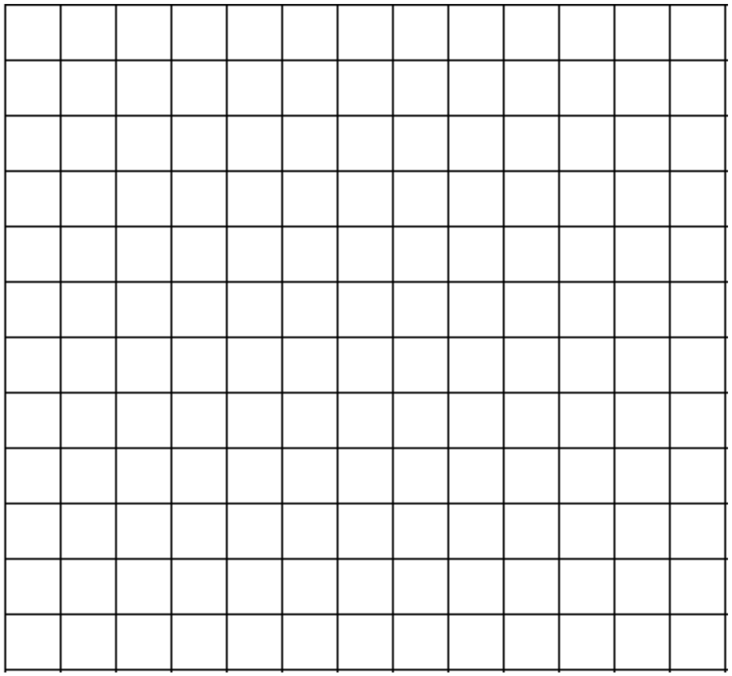
The three regular tilings of the Euclidean plane.

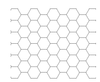
Definition:

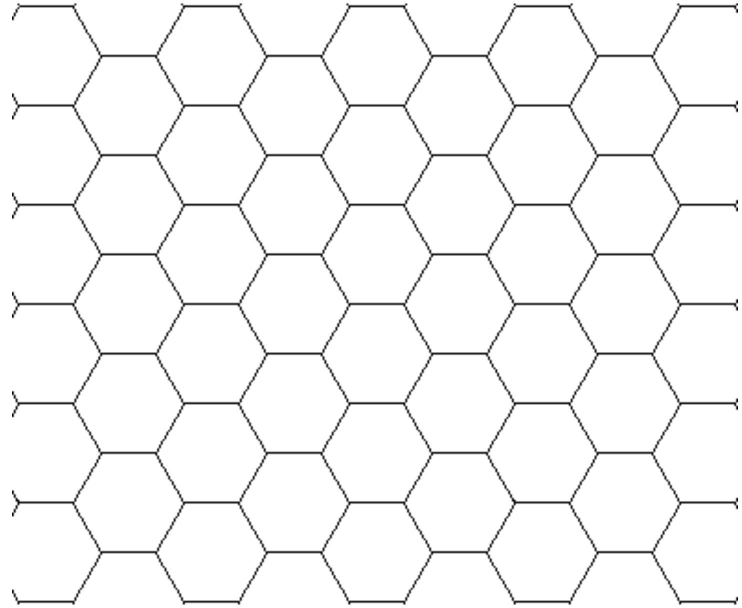
An (n,k) tiling is a regular tiling by n-sided polygons, meeting k to a vertex.

This is the (3,6) tiling of the plane.

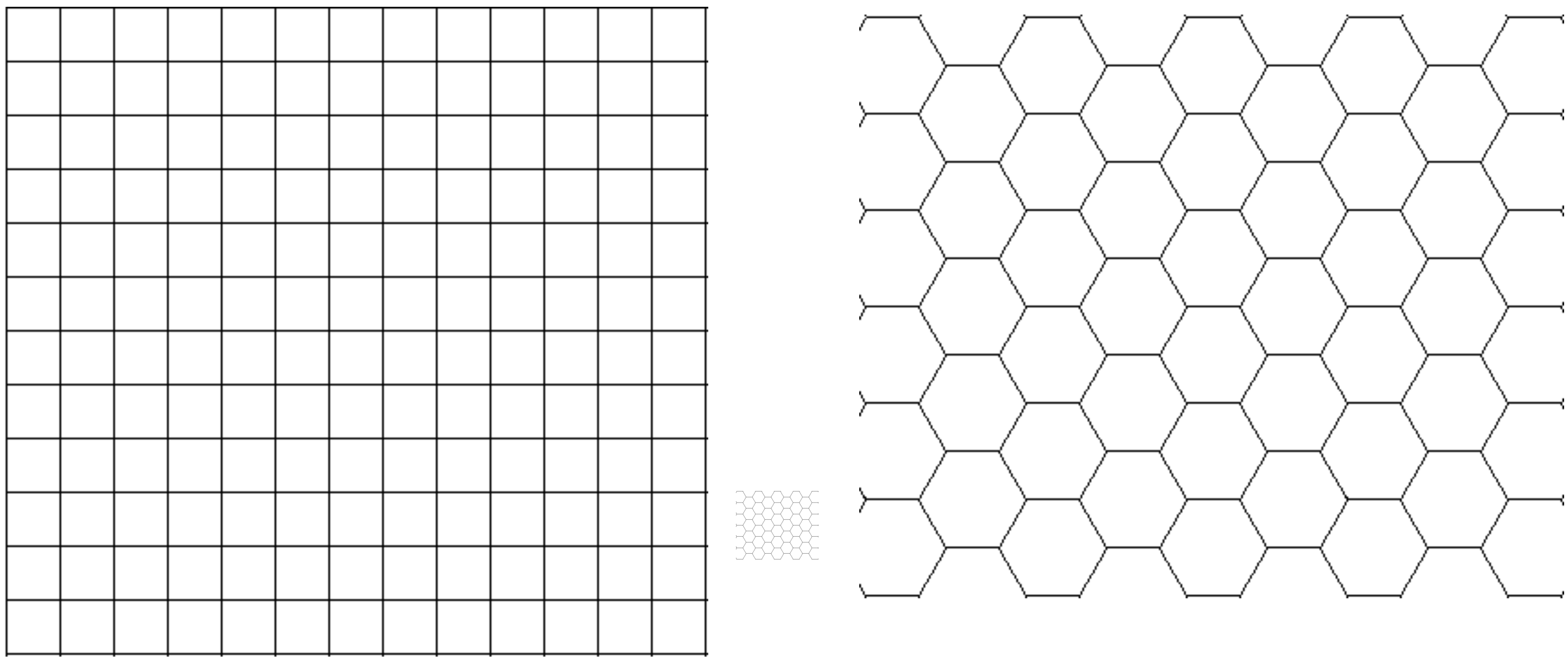








Which tilings are these?



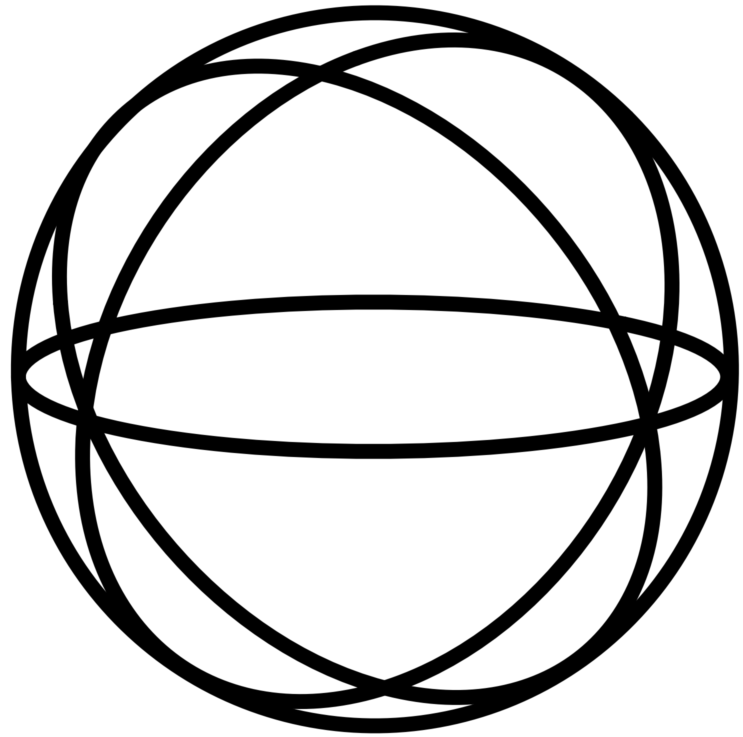
The (4,4) tiling and the (6,3) tiling.

Tilings on the Sphere

From last time:

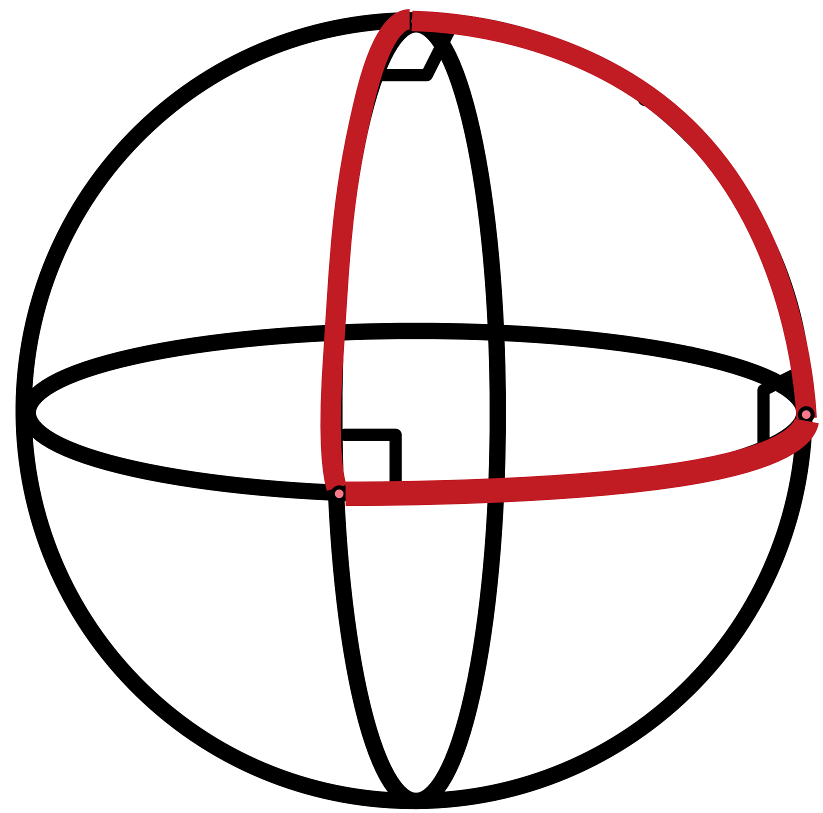
On the sphere, lines are great circles.

Triangles have angle sum bigger than 180°.



Example:

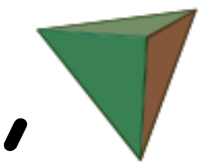
we can use this triangle to make a (3,4) tiling of the sphere.



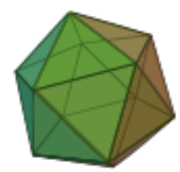
Platonic Solids

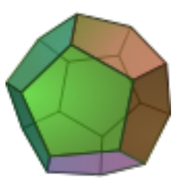
Use the paper cutouts to make these polyhedra:

 cube,

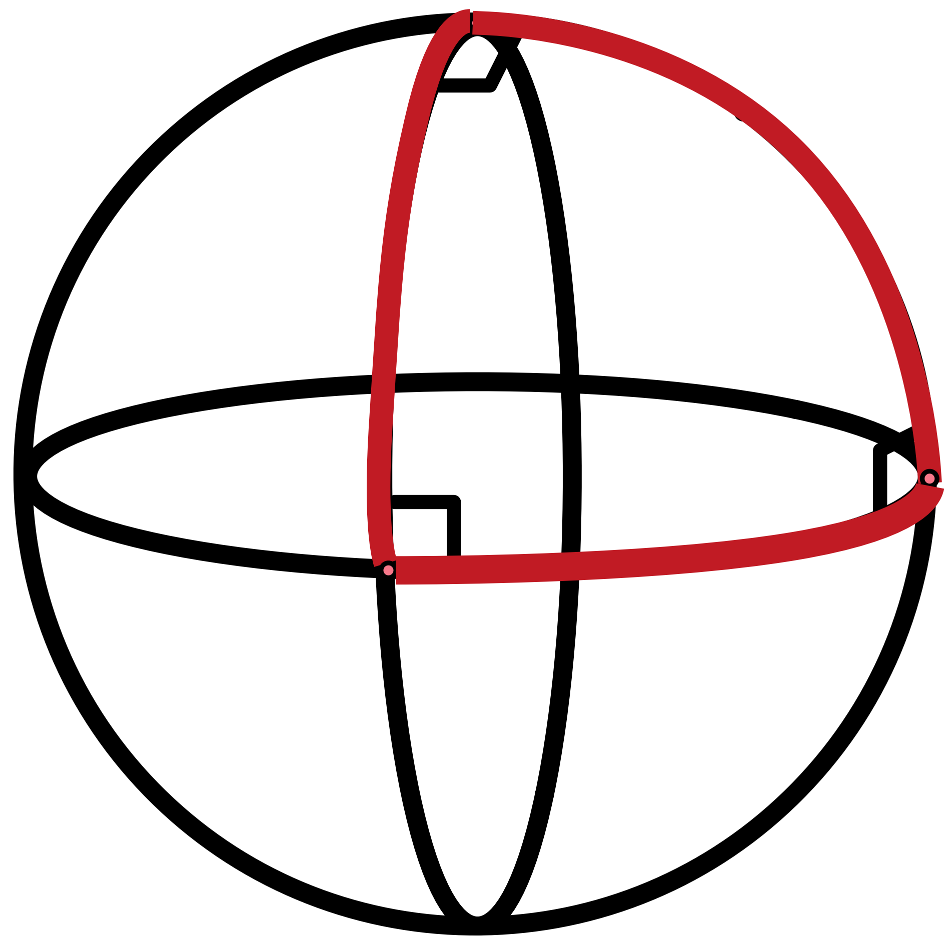
tetrahedron, 

 octahedron,

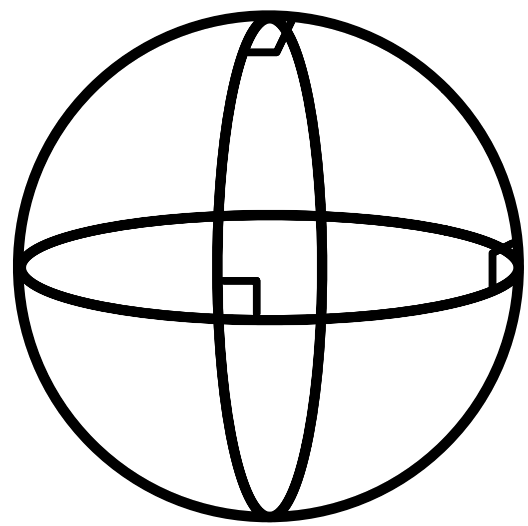
icosahedron, 

 dodecahedron

Which polyhedron matches the (3,4) tiling?



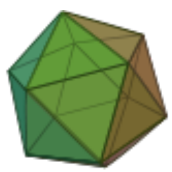
The octahedron matches the (3,4) tiling.



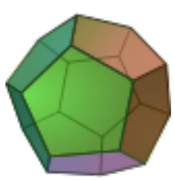
Problem: Which tilings do these polyhedra match?


tetrahedron


octahedron


icosaherdon


cube

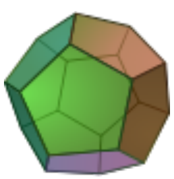

dodecahedron

 tetrahedron: (3,3) tiling

 octahedron: (3,4) tiling

 icosahedron: (3,5) tiling

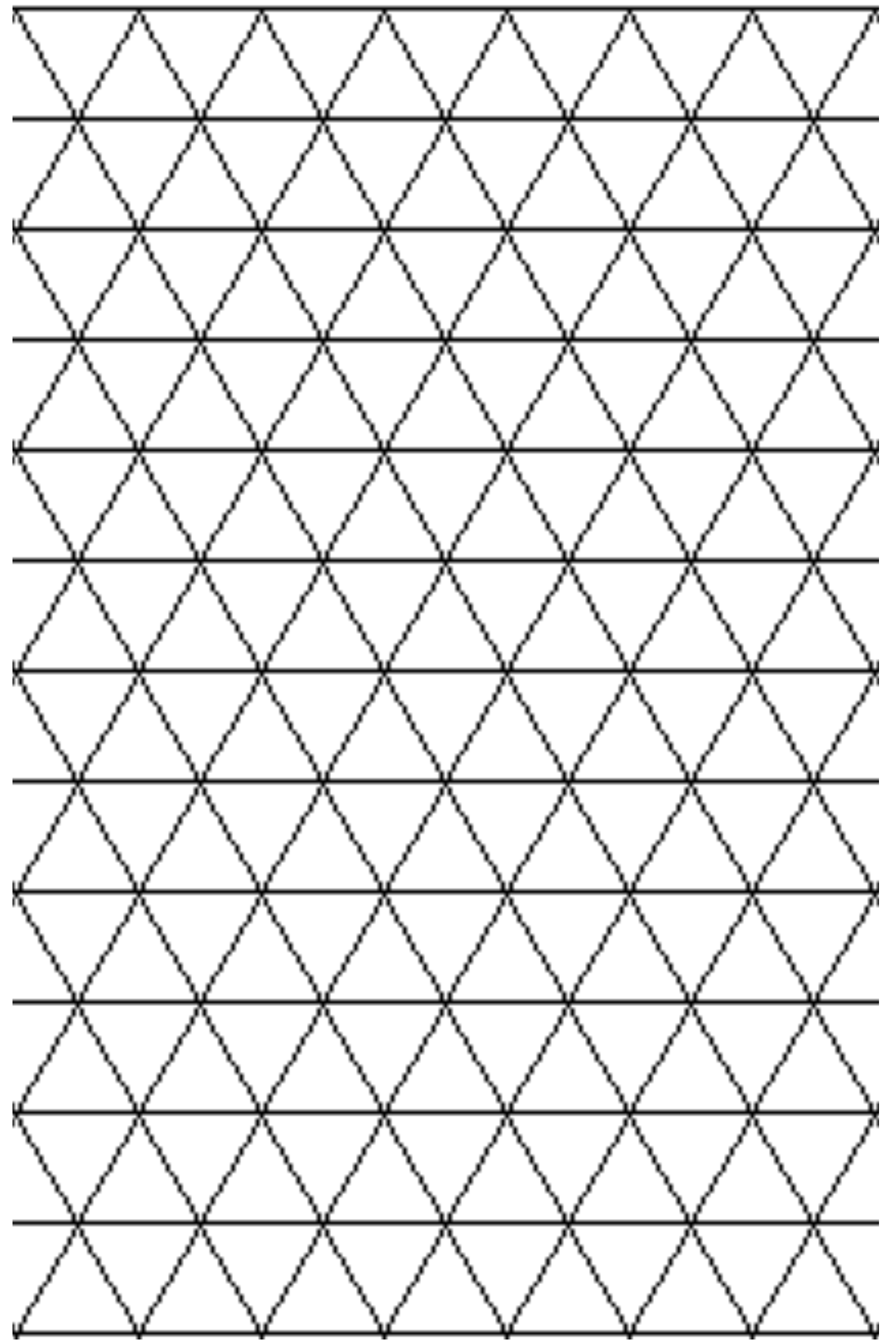
 cube: (4,3) tiling

 dodecahedron: (5,3) tiling

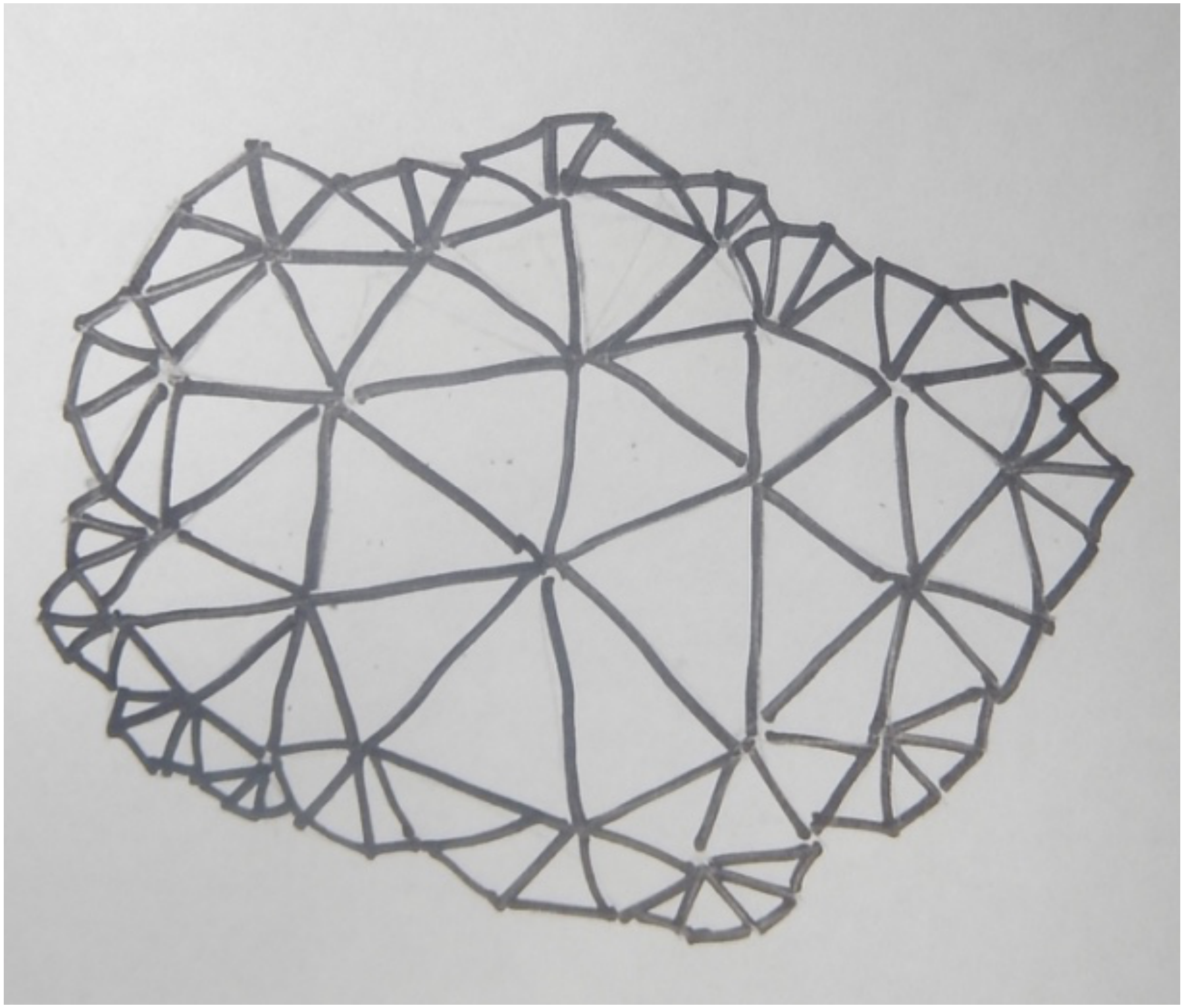
If we put 3, 4, or 5 regular triangles around each vertex, we get a tiling of the SPHERE.

What happens if we put 6?

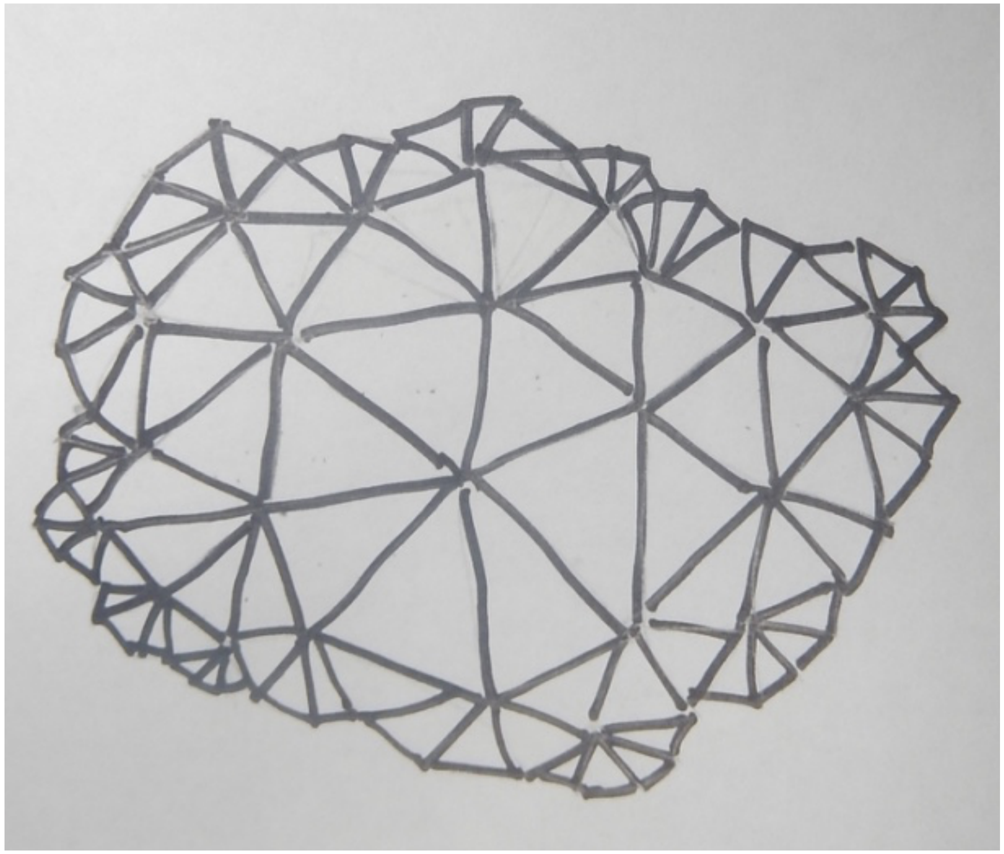
The (3,6) tiling fills the Euclidean plane.



What happens if we put 7 regular triangles at each vertex?



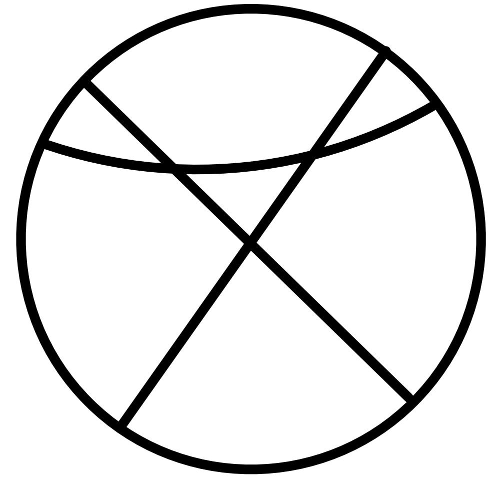
Here is a hand drawn sketch of a bit of a (3,7) tiling.

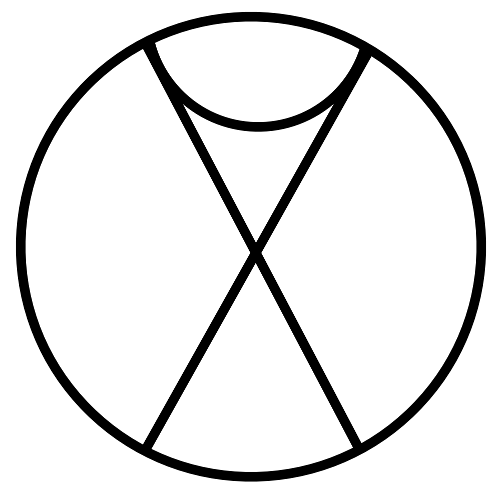


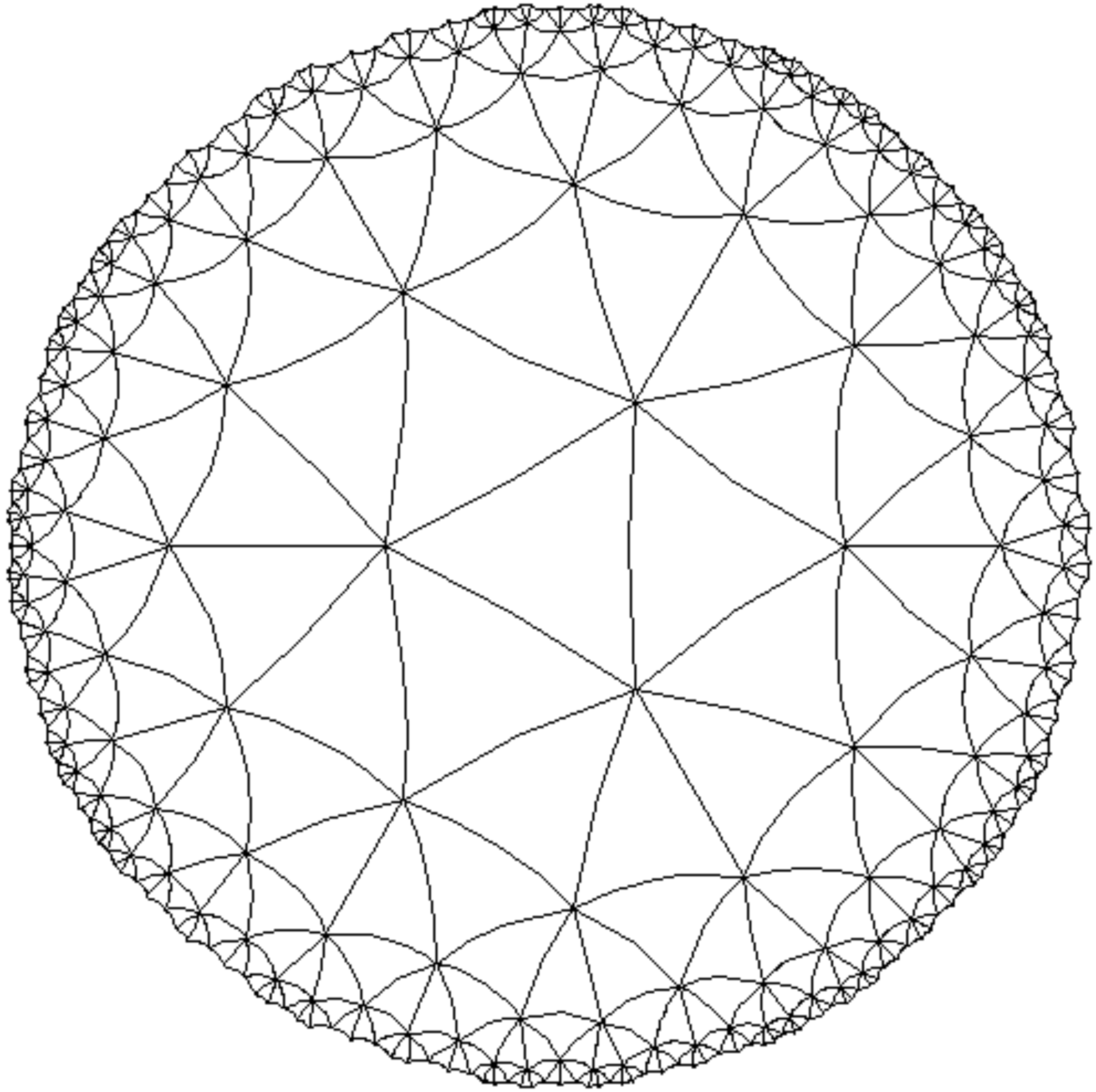
What kind of geometry does this look like?

From last time:

On the hyperbolic plane, triangles have angle sum less than 180°.



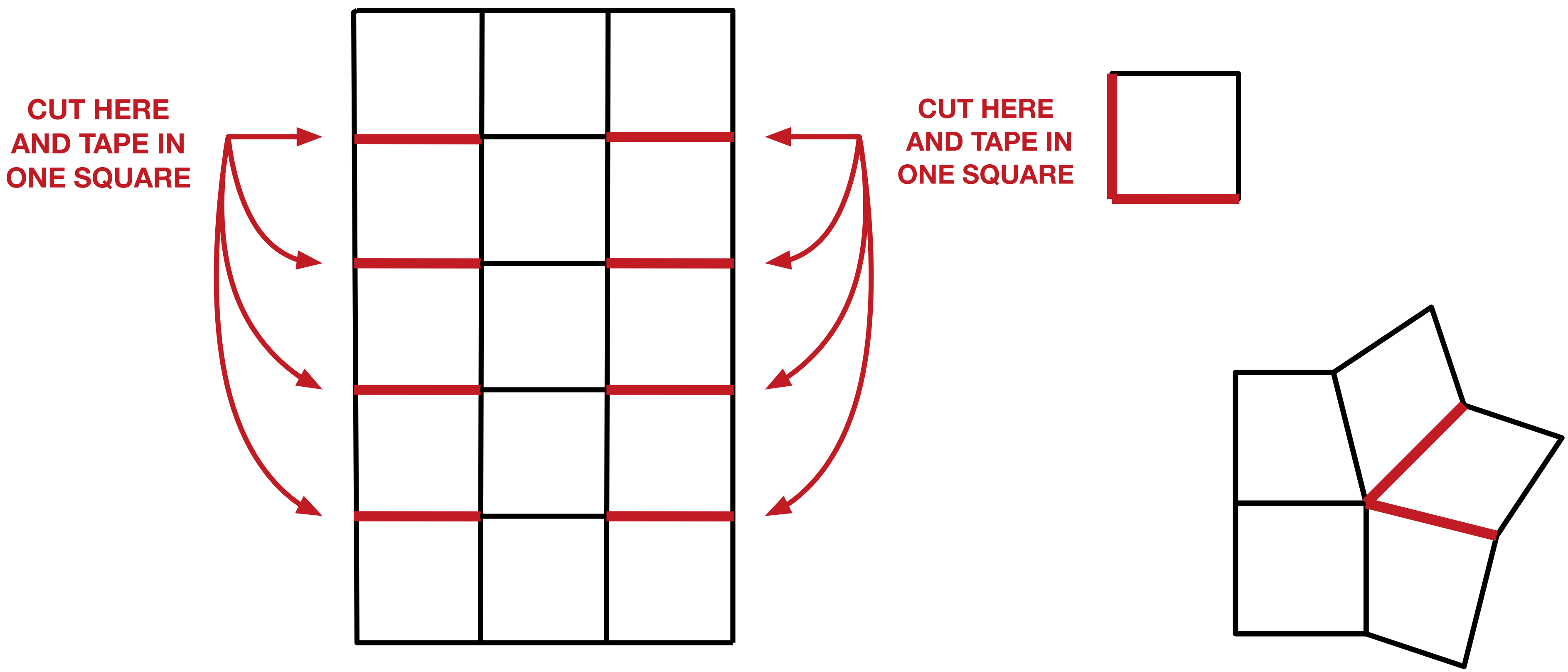




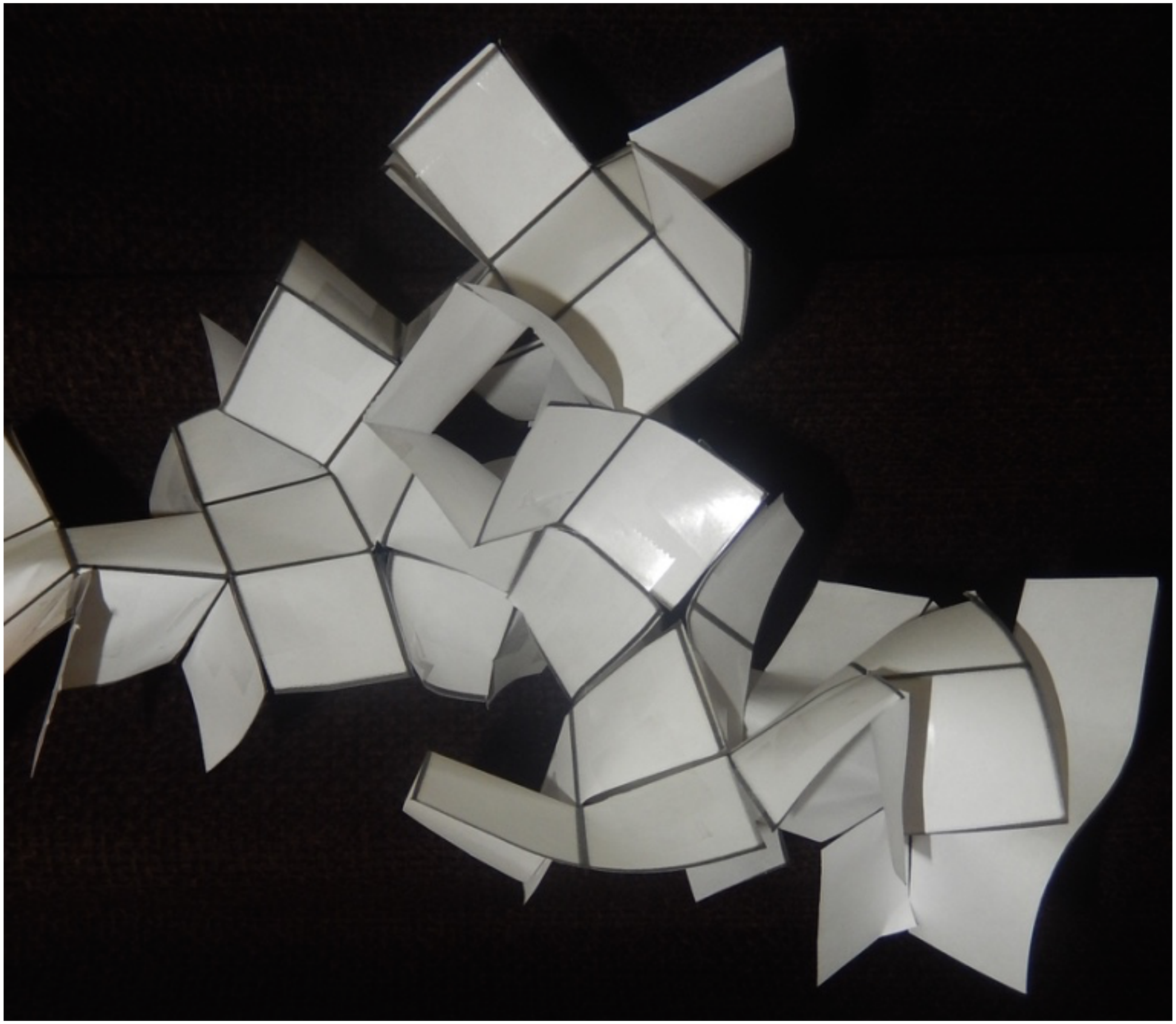
Picture credit: Keith Conrad

Here is a (3,7) tiling.

Can we put the (4,5) tiling into the hyperbolic plane?

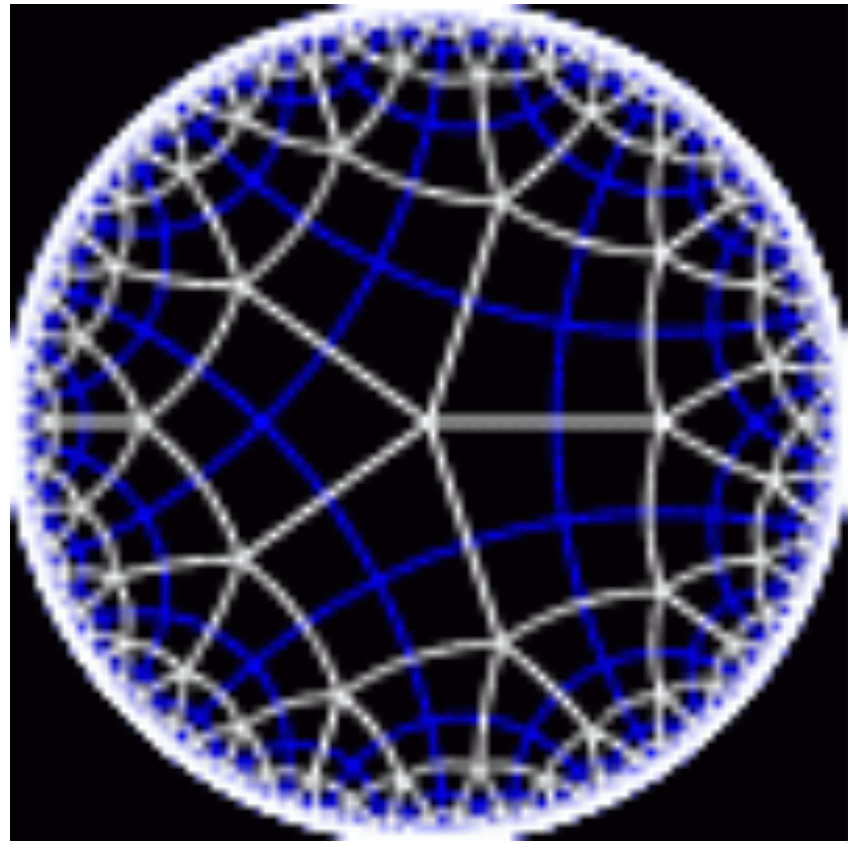


You can make a piece of the (4,5) tiling with paper and tape.



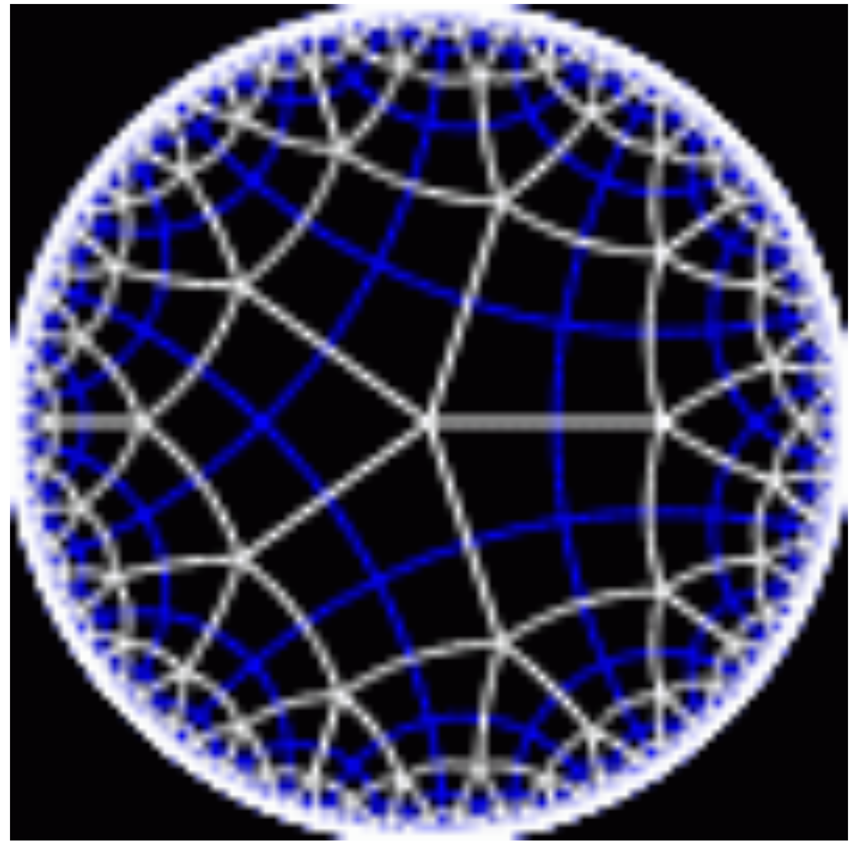
A piece of a (4,5) tiling.

We can tile hyperbolic space with a (4,5) tiling.



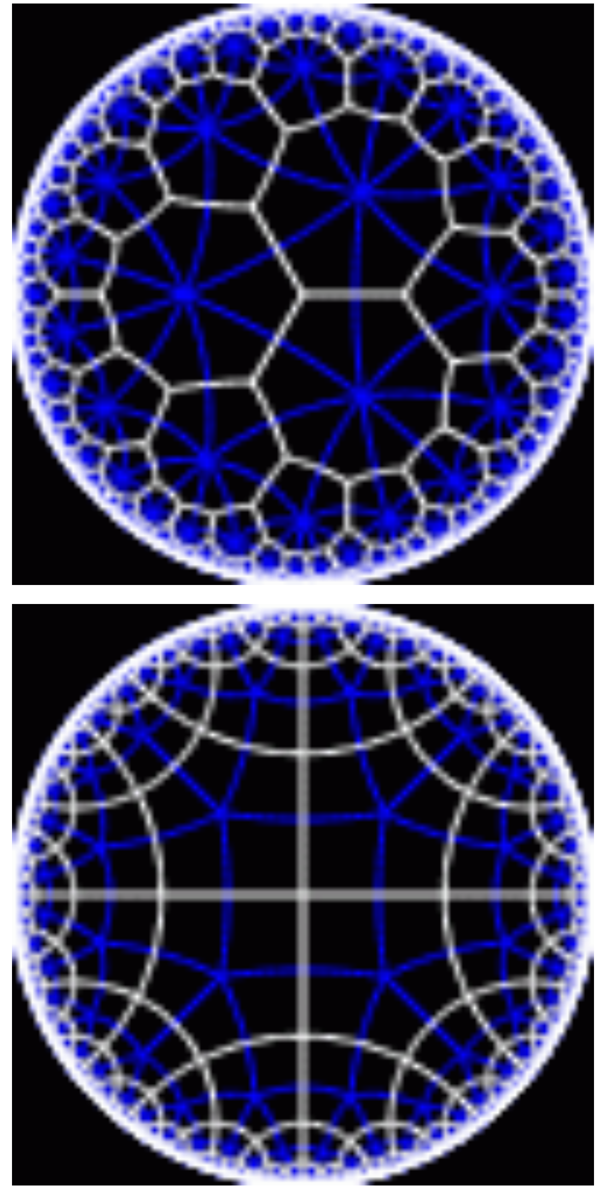
Picture: Don Hatch

Notice how lines that start near each other end up far apart.



Picture: Don Hatch

More examples: Here are the (8,3) and (5,4) tilings of hyperbolic space.



Picture: Don Hatch

Problem:

Find a simple formula in n and k that tells you if the (n,k) tiling fits on the sphere, the Euclidean plane, or the hyperbolic plane.

Data:

Spherical: (3,3) (3,4), (3,5), (4,3), (5,3)

Euclidean: (3,6), (4,4), (6,3)

Hyperbolic: (3,7), (4,5), (5,4), (8,3), and lots of others

An (n,k) tiling is

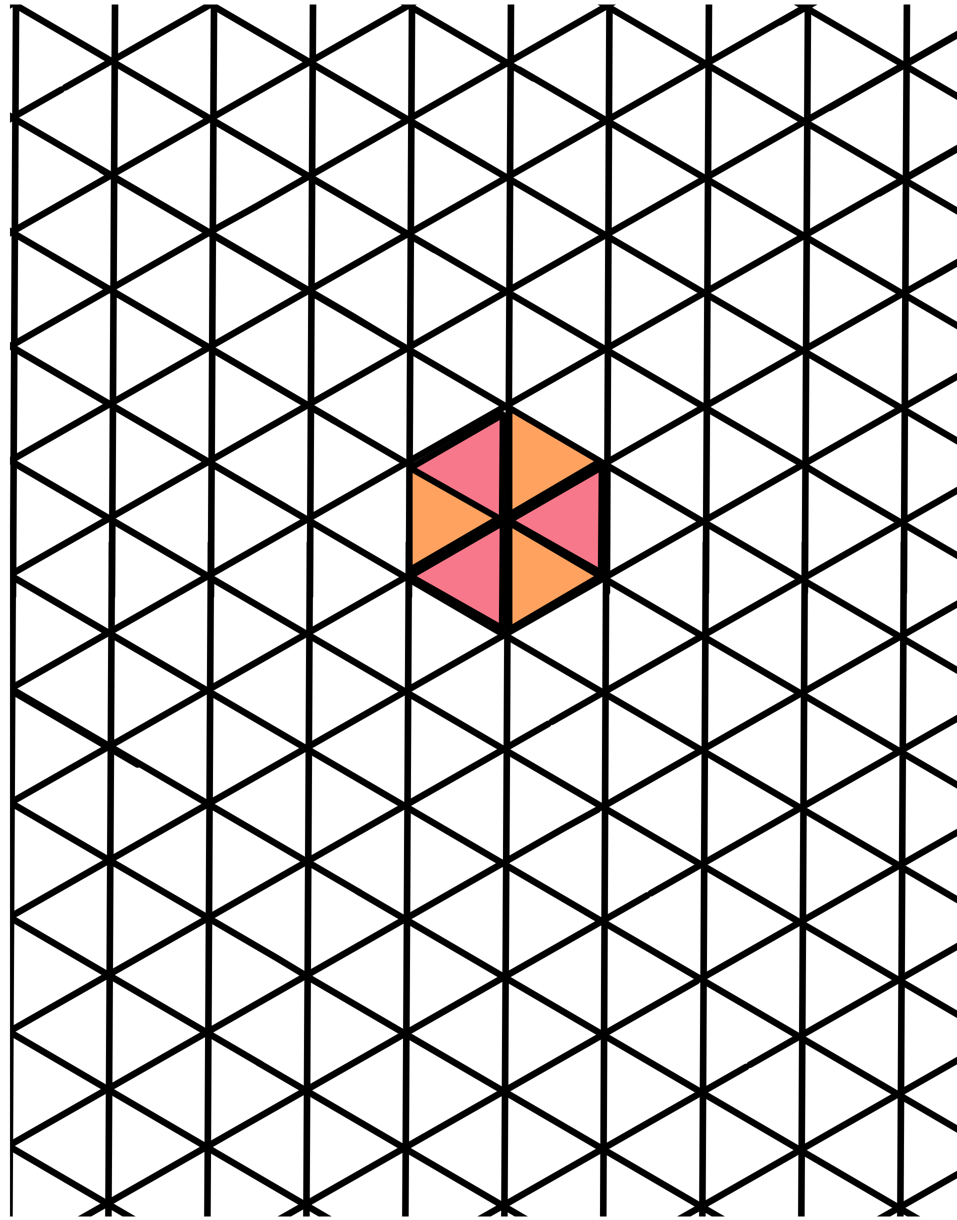
Spherical if $1/n + 1/k > 1/2$

Euclidean if $1/n + 1/k = 1/2$

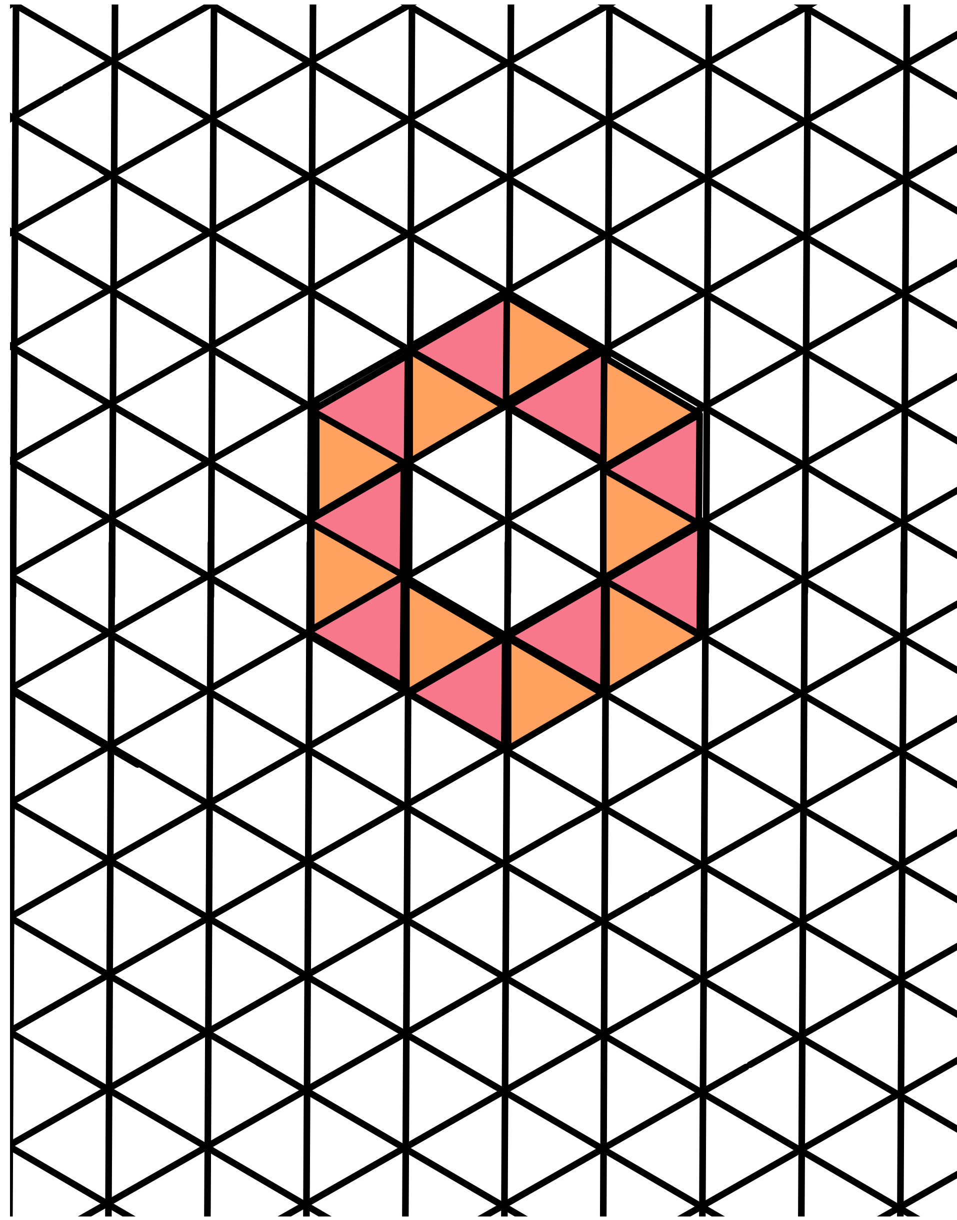
Hyperbolic if $1/n + 1/k < 1/2$.

Growth Rates

In Euclidean space, the disk of radius 1 has 6 triangles.

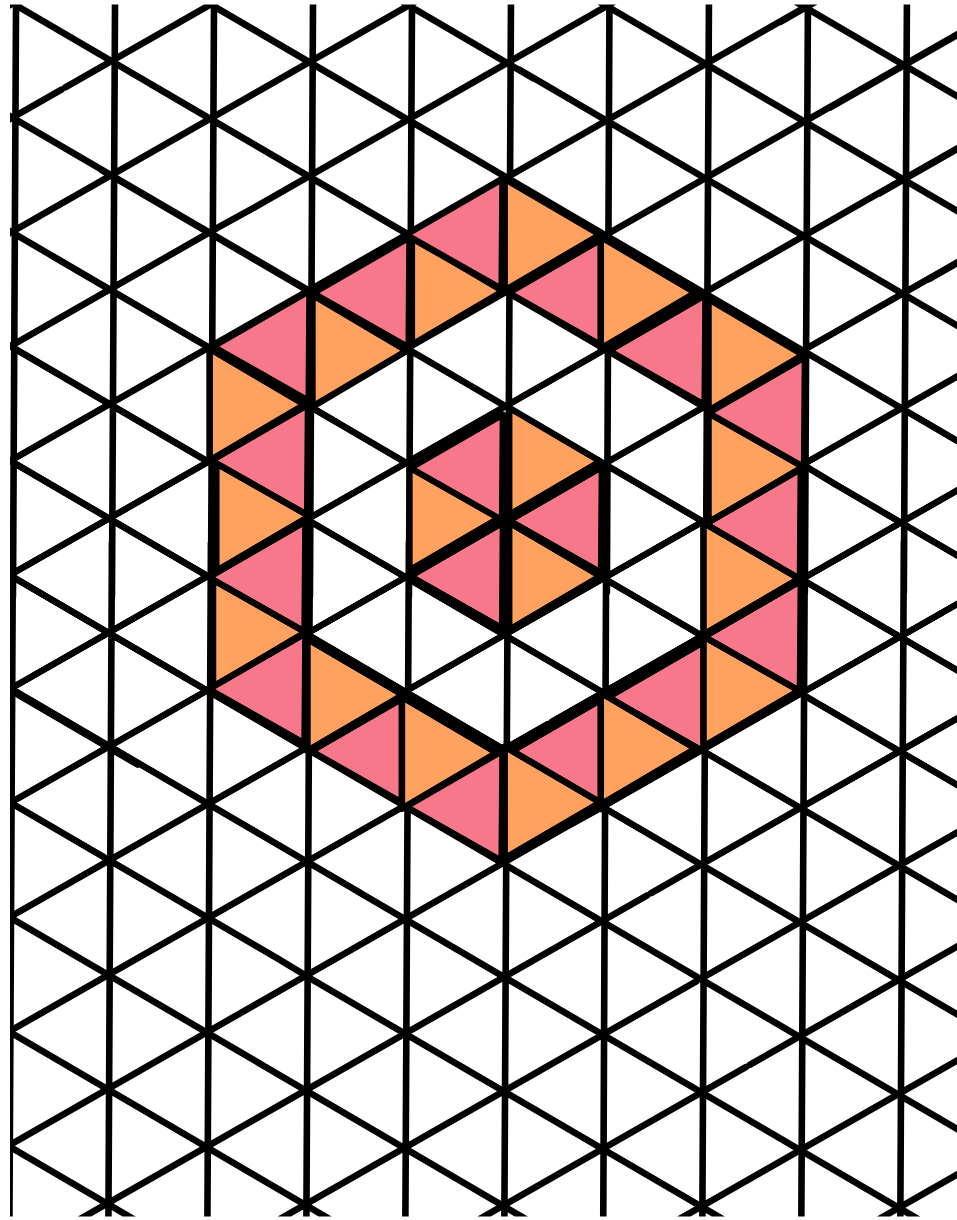


The disk with radius 2 has $6 + 18 = 24$ triangles altogether.



Problem:

How many triangles are in the disk of radius 3?



Problem:

Find a formula for the number of triangles in a disk of radius R.

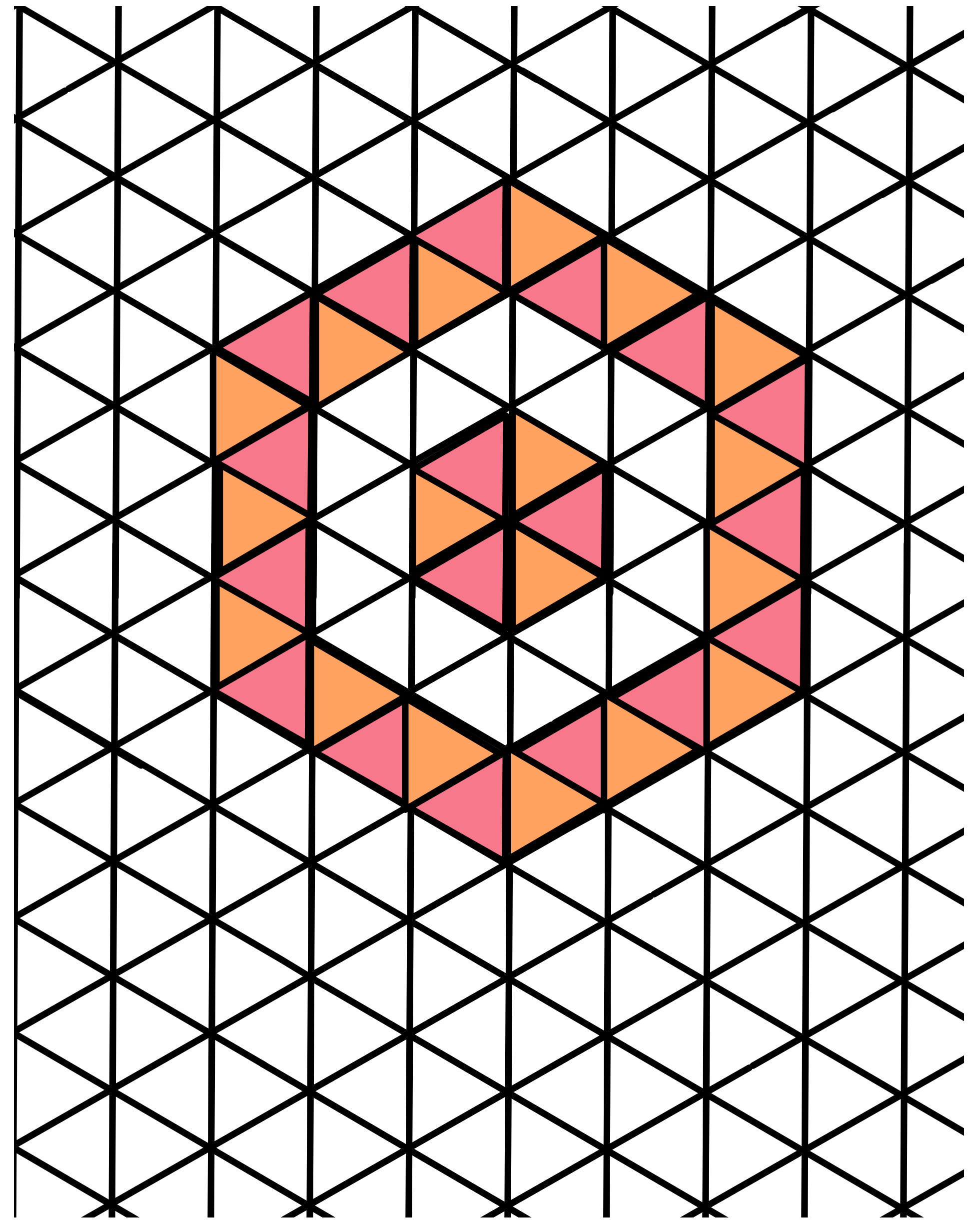
radius 1: 6

radius 2: 24

radius 3: 54

radius 4: 96

radius 5: 150



radius 1: 6

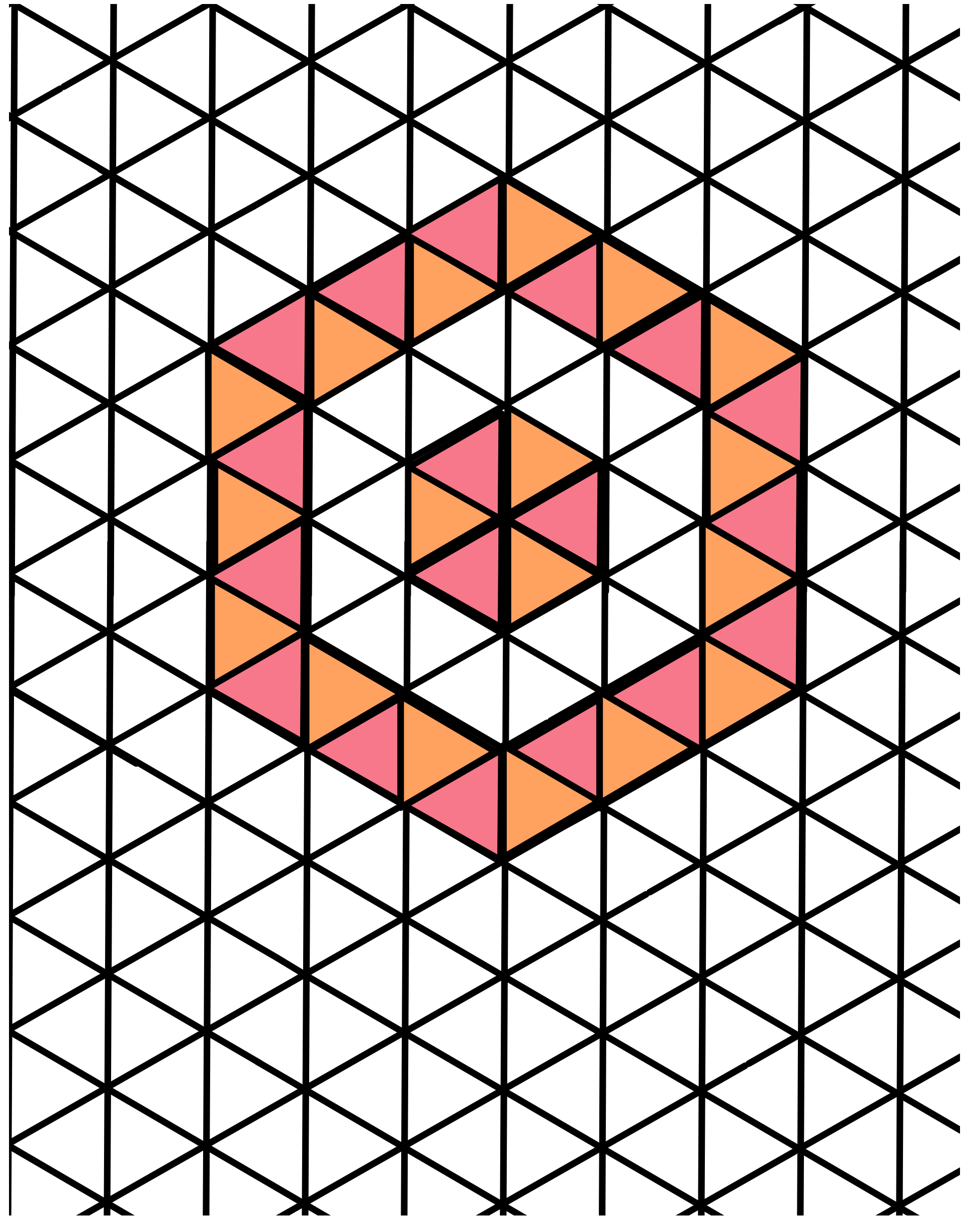
radius 2: 24

radius 3: 54

radius 4: 96

radius 5: 150

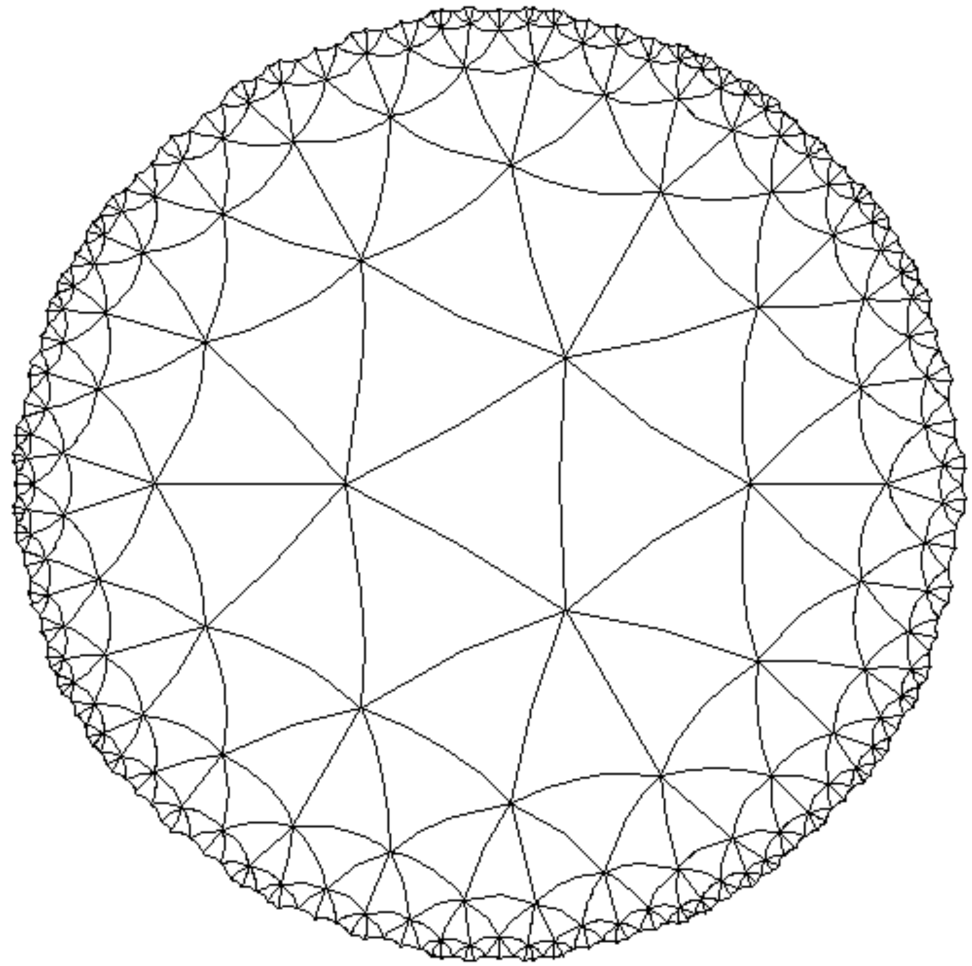
radius R: $6R^2$



Hyperbolic Tilings

Now let's look at the (3,7) tiling.

Again, we count the number of triangles in disks of radius 1, 2, 3, and so on.



Picture credit: Keith Conrad

Radius 1: 7

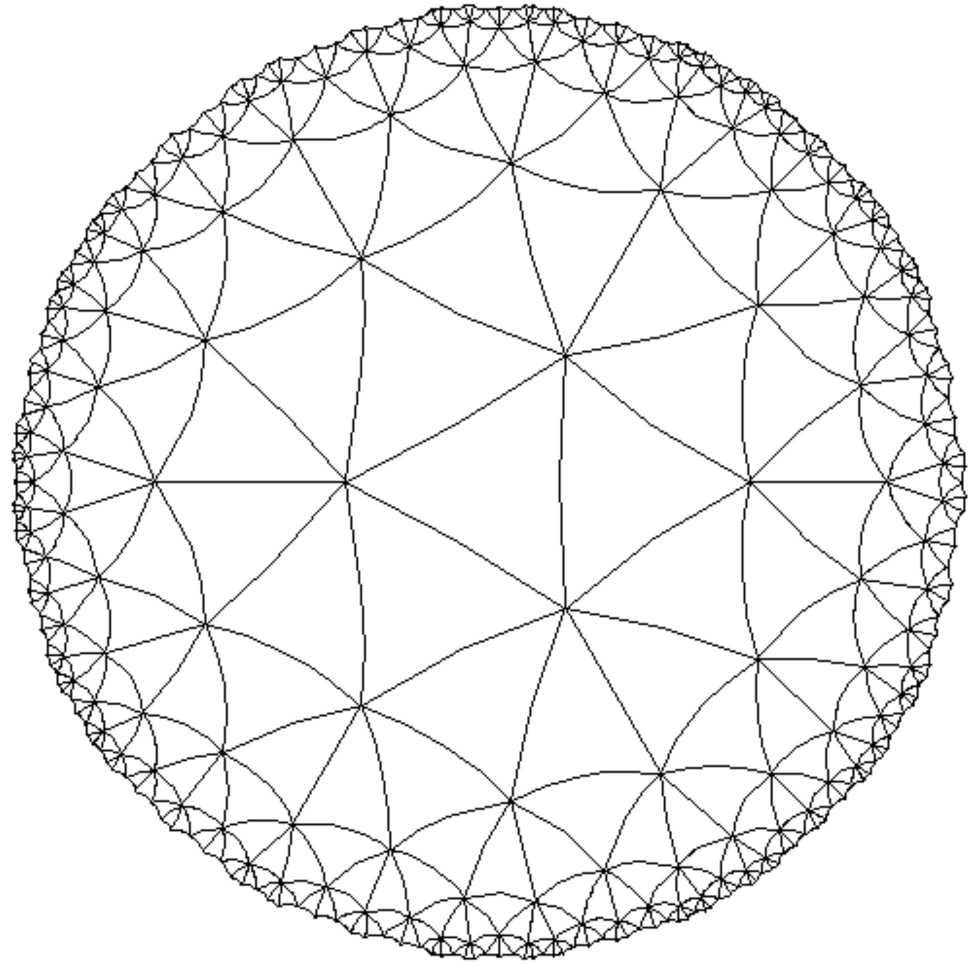
Radius 2: 35

Radius 3: 112

Radius 4: 315

Radius 5: 847

Radius 6: 2240



Picture credit: Keith Conrad

A radius 10 disk in the (3,6) Euclidean tiling has 600 triangles.

A radius 10 disk in the (3,7) hyperbolic tiling has 105875 triangles.

On the hyperbolic plane, the area inside a circle grows much faster — more like 3^R than R^2.

Perhaps this is why some plants and animals look hyperbolic — more surface area means more nutrients.

Puzzle:

Find a recurrence describing the number of triangles in each ring.

Ring 1: 7

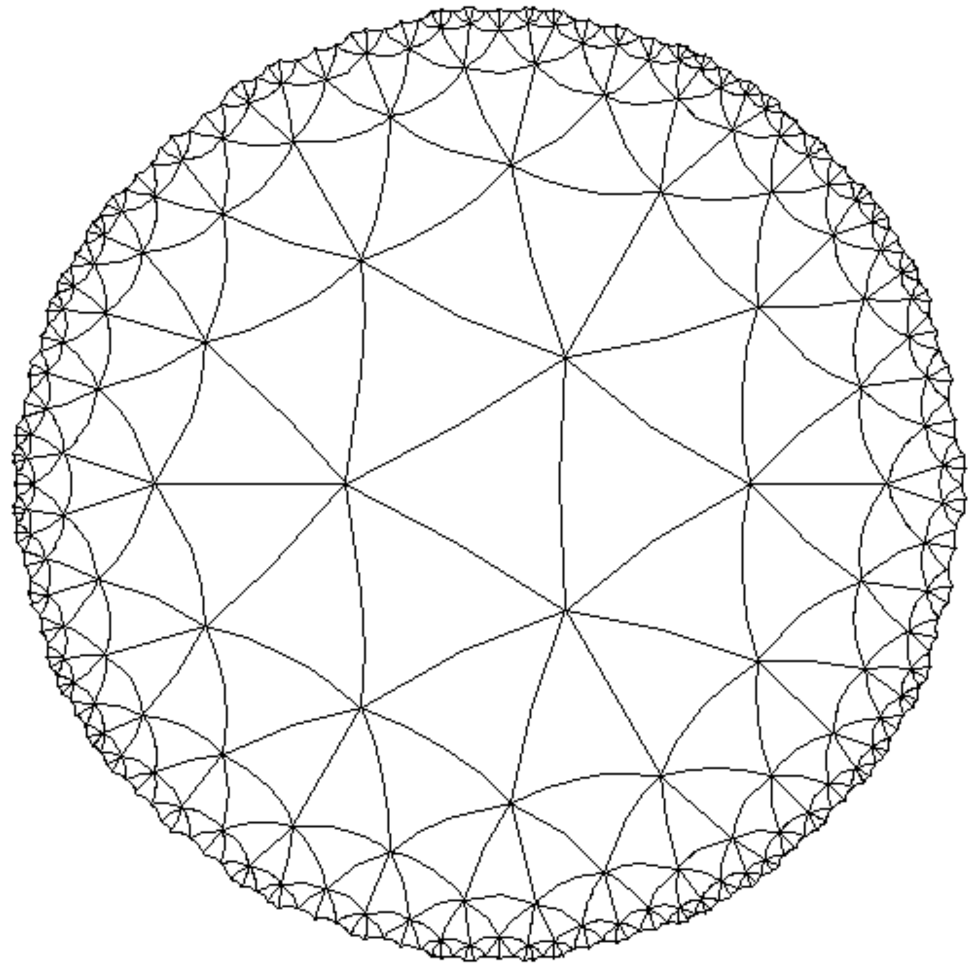
Ring 2: 28

Ring 3: 77

Ring 4: 203

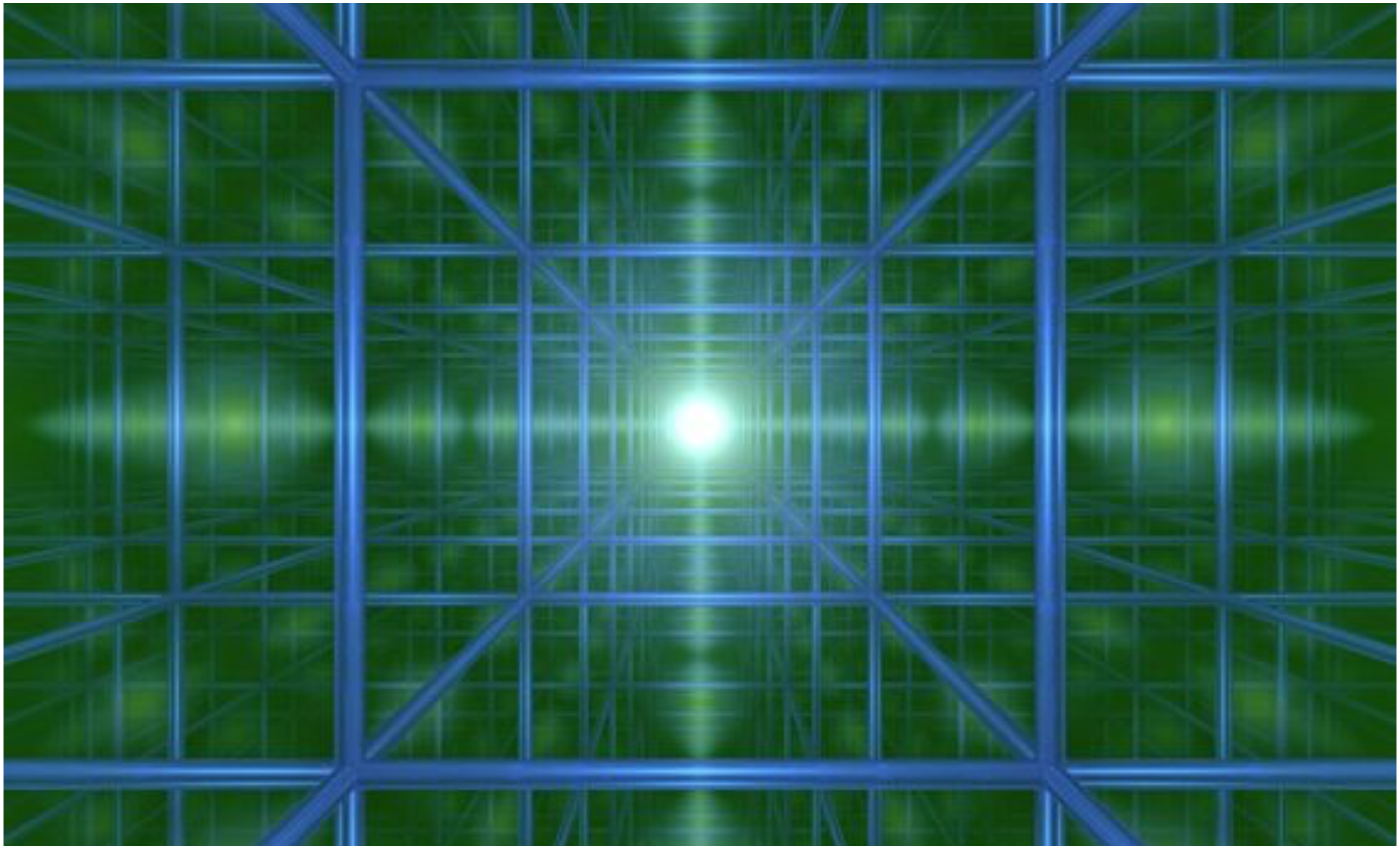
Ring 5: 532

Ring 6: 1393

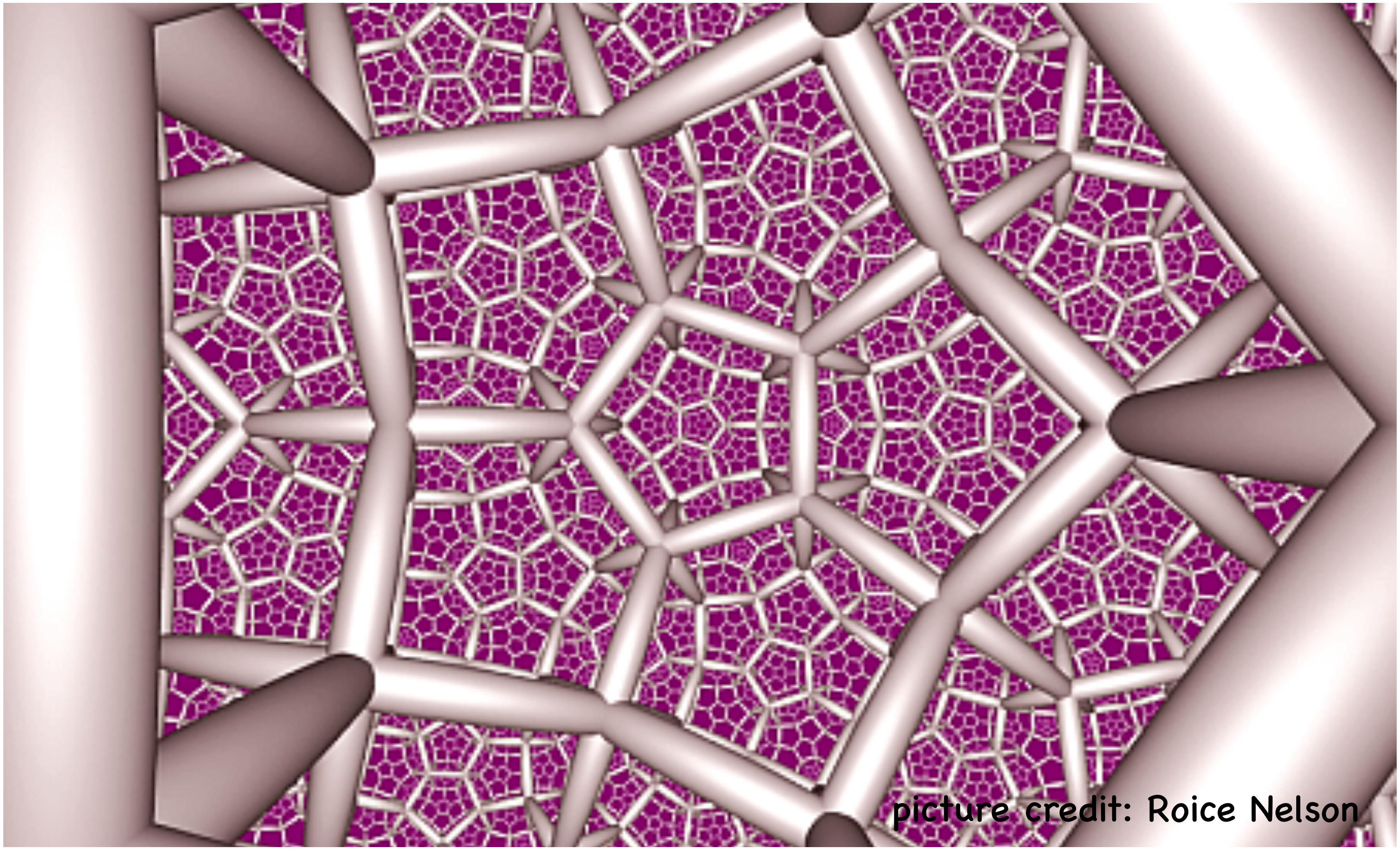


Picture credit: Keith Conrad

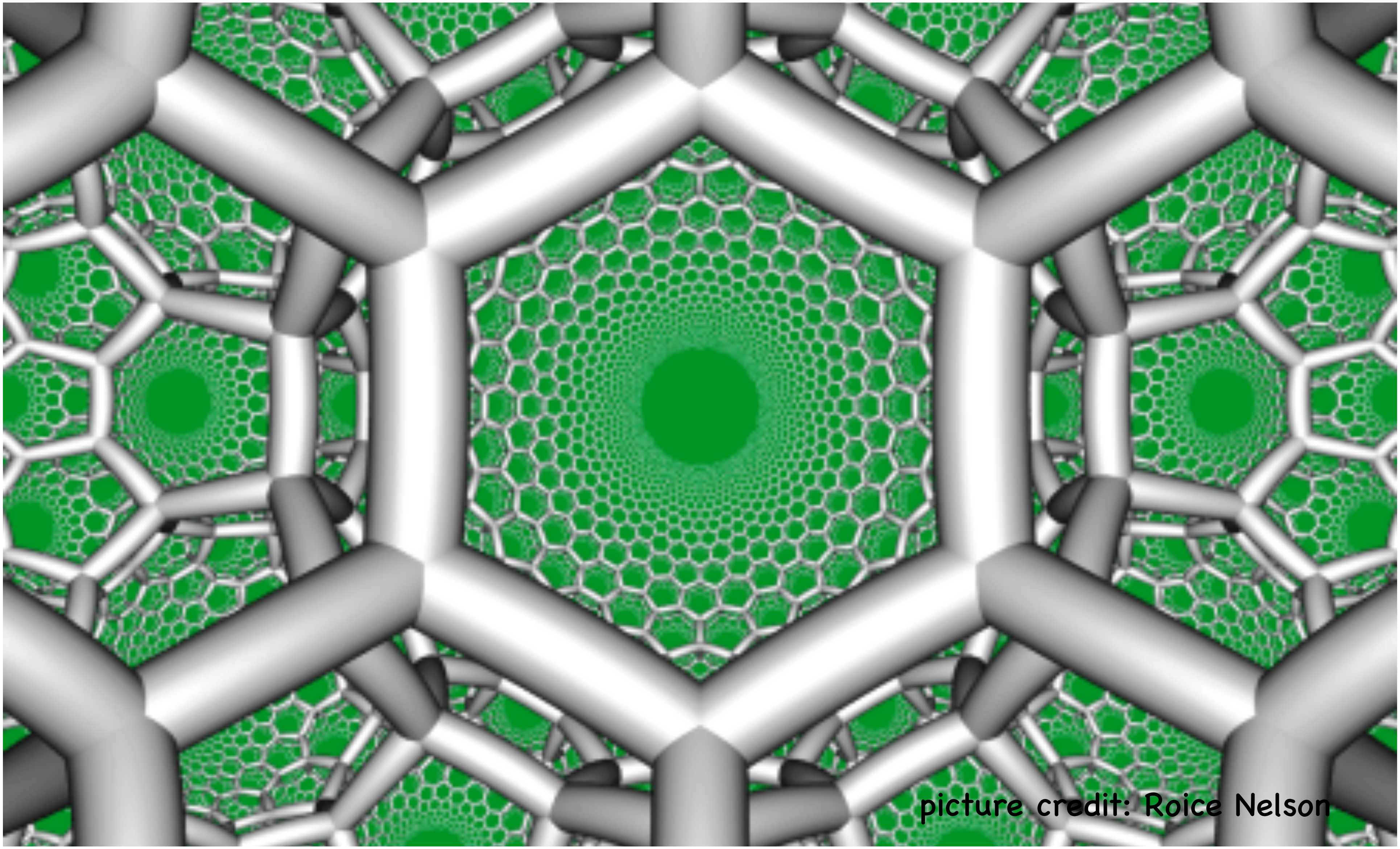
Tilings in 3 dimensions



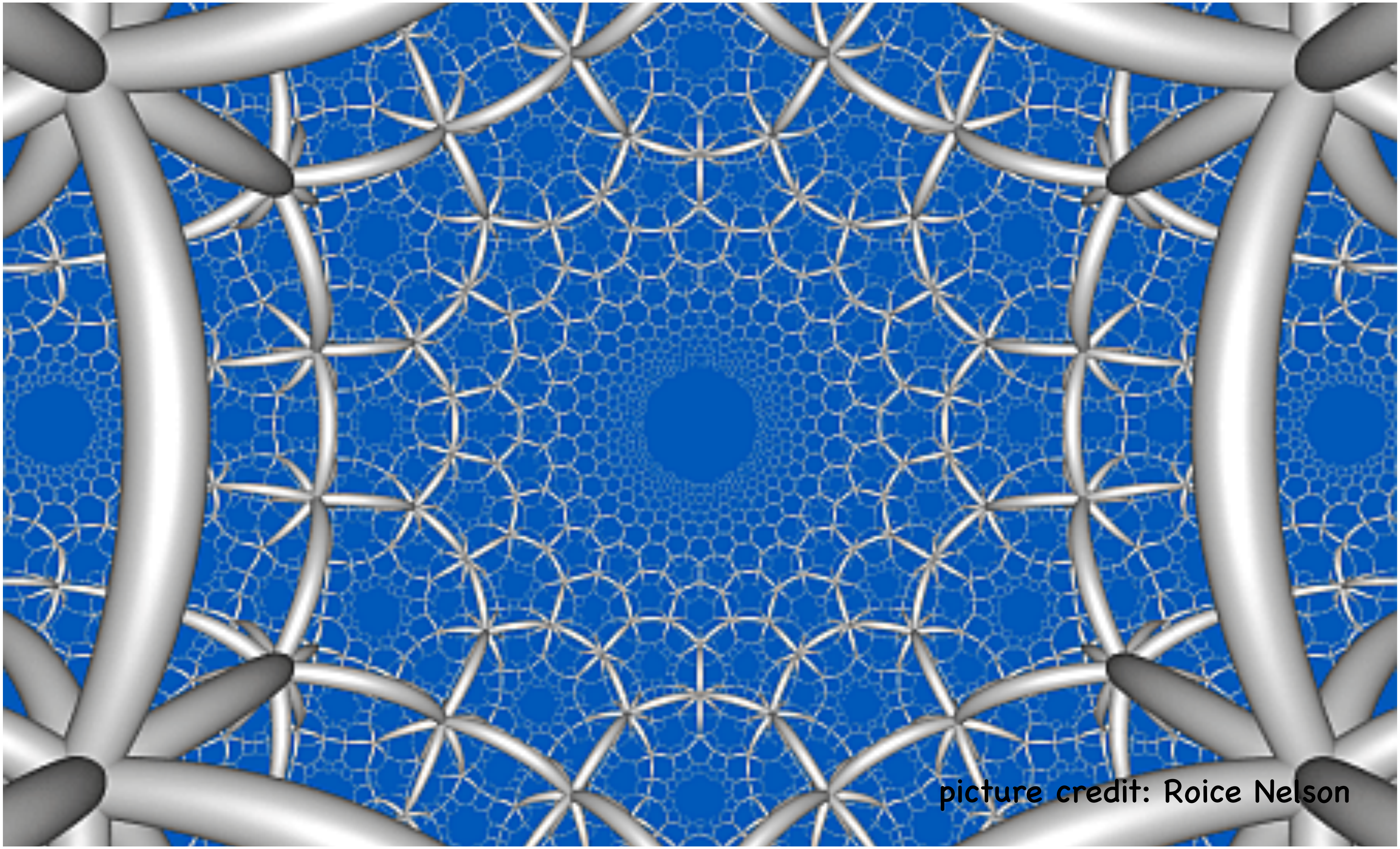
(4,3,4) Euclidean tiling – 4 cubes meet around each edge.



(5,3,4) hyperbolic tiling by dodecahedrons



(6,3,3) tiling



(6,3,5) tiling

شكرا جزيلا !

Some cool links

1. https://www.youtube.com/watch?v=YzzJGeiucNg&nohtml5=False

2. https://johncarlosbaez.wordpress.com/2014/05/14/hexagonal-hyperbolic-honeycombs/

3. http://www.plunk.org/~hatch/HyperbolicTesselations/

4. http://www.josleys.com/show_gallery.php?galid=325

5. https://www.youtube.com/watch?v=p7HB2cfZ4mw