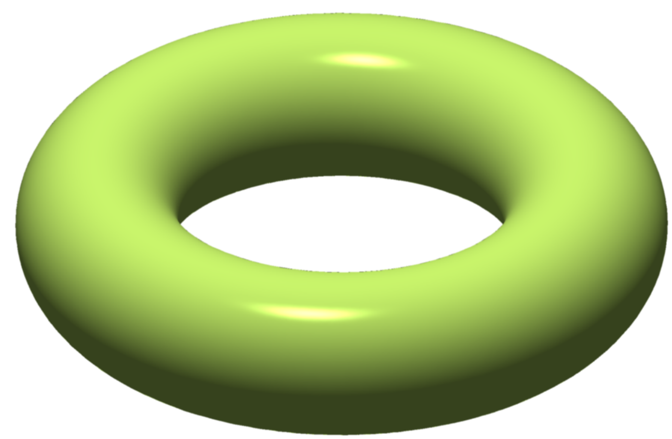
SARSI 2016
First Week Lectures
Math – Kim Whittlesey

Lecture 3
Geometry of Surfaces
هندسة السطوح

Surfaces

A torus
looks like the
skin of a donut.

The torus is
the surface,
not the inside.



A genus 2 torus or "double" torus has two handles.

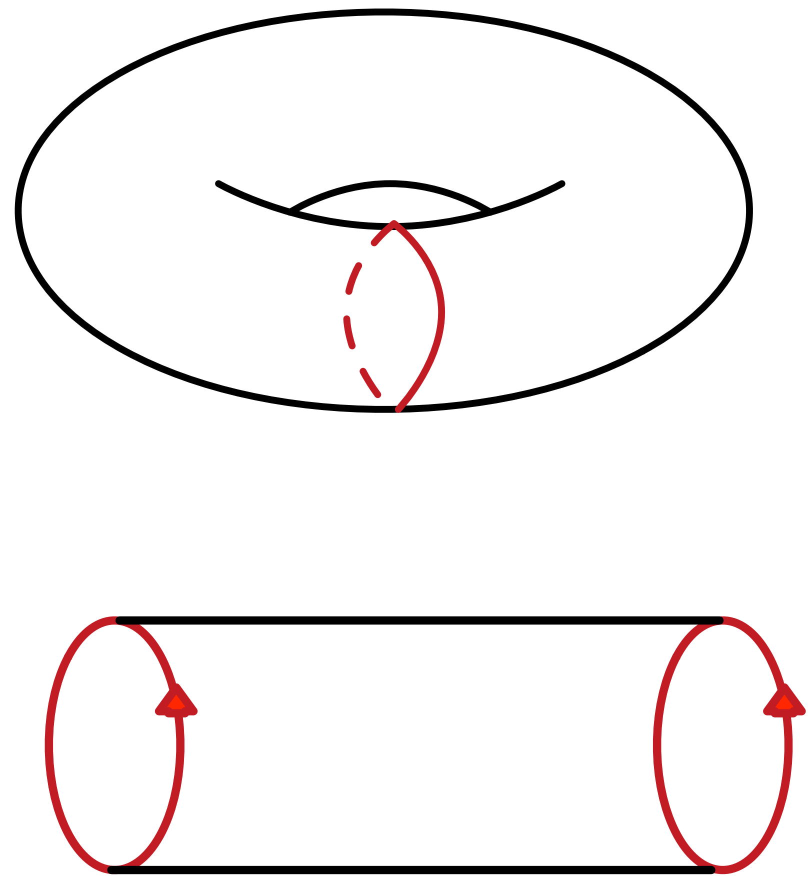
The genus is the number of handles.

The skin of this pretzel has genus 3.

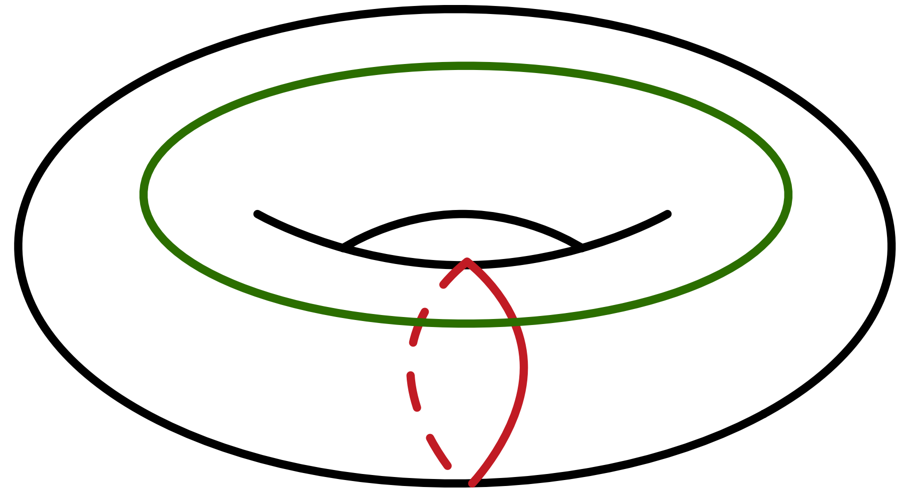
The sphere is also a surface.

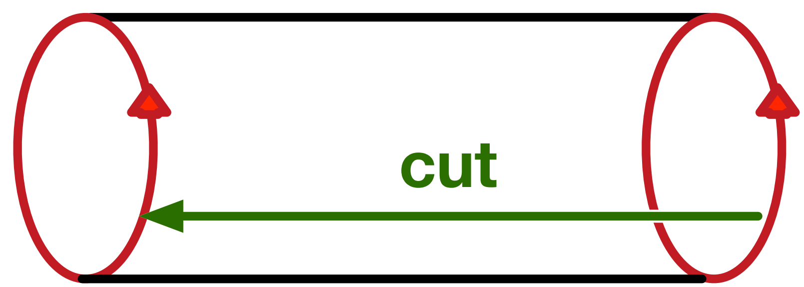
The sphere is also a surface. It has genus 0.

If you cut a torus along a circle through the hole, you get a cylinder.

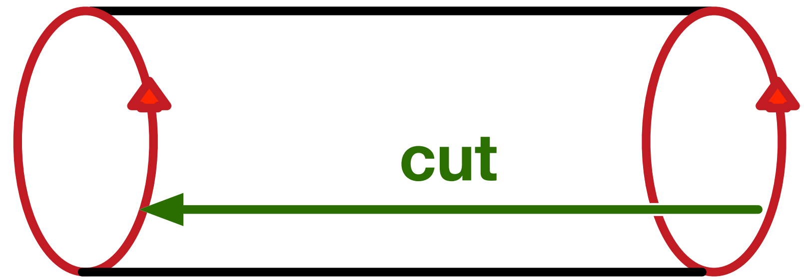


What do you get if you cut the torus along both of the curves shown?

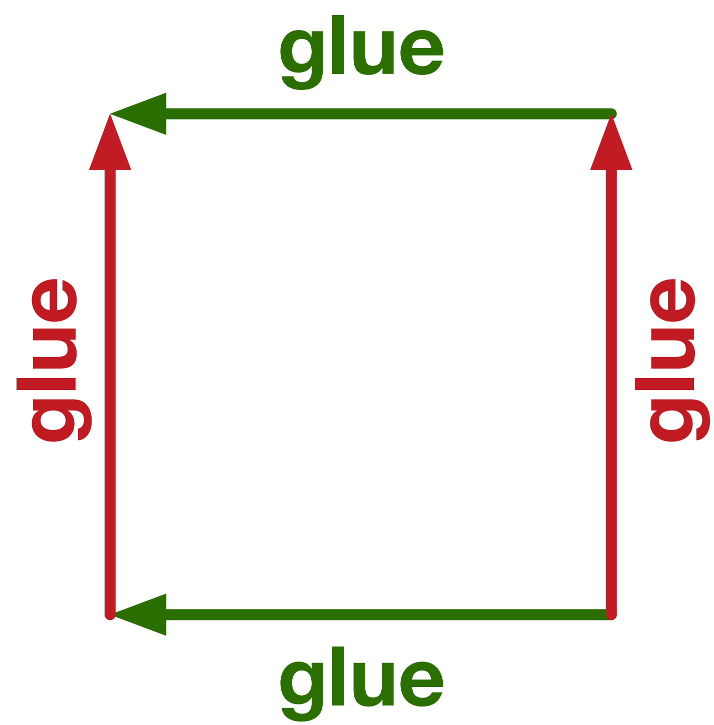




You get a rectangle.



To get the torus, glue opposite sides together without twisting.

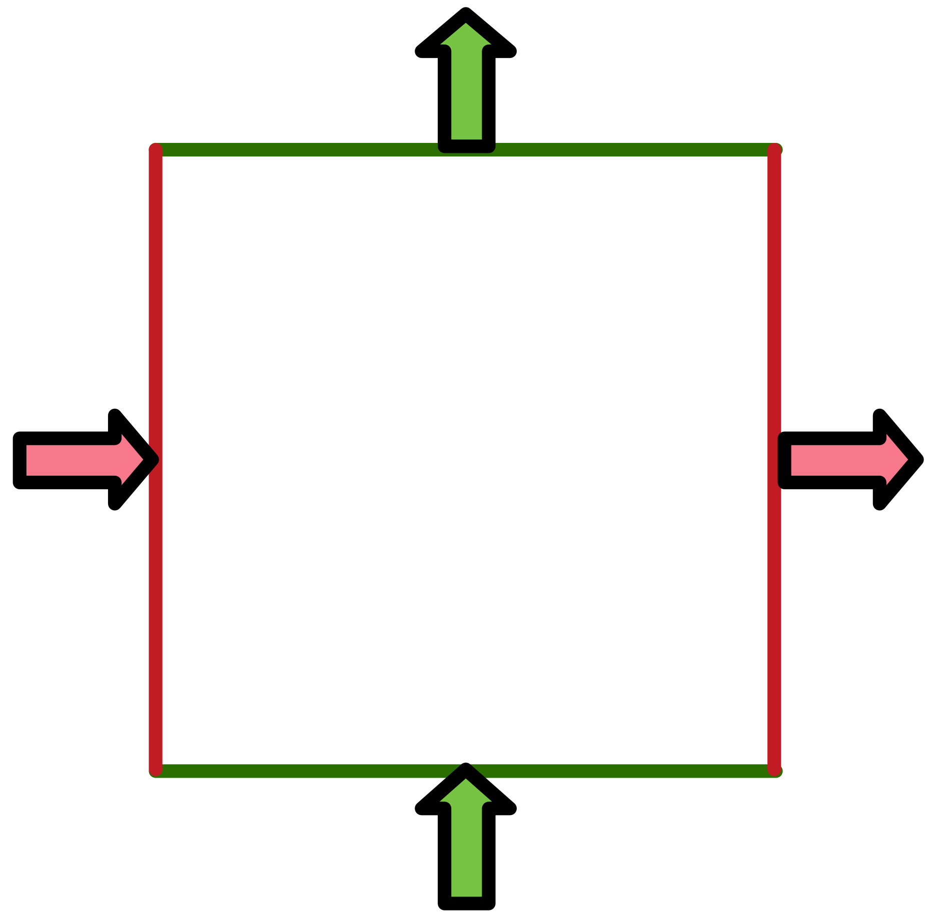


If try to make a torus out of a square of paper, it won't look the same, since paper is not very stretchy.

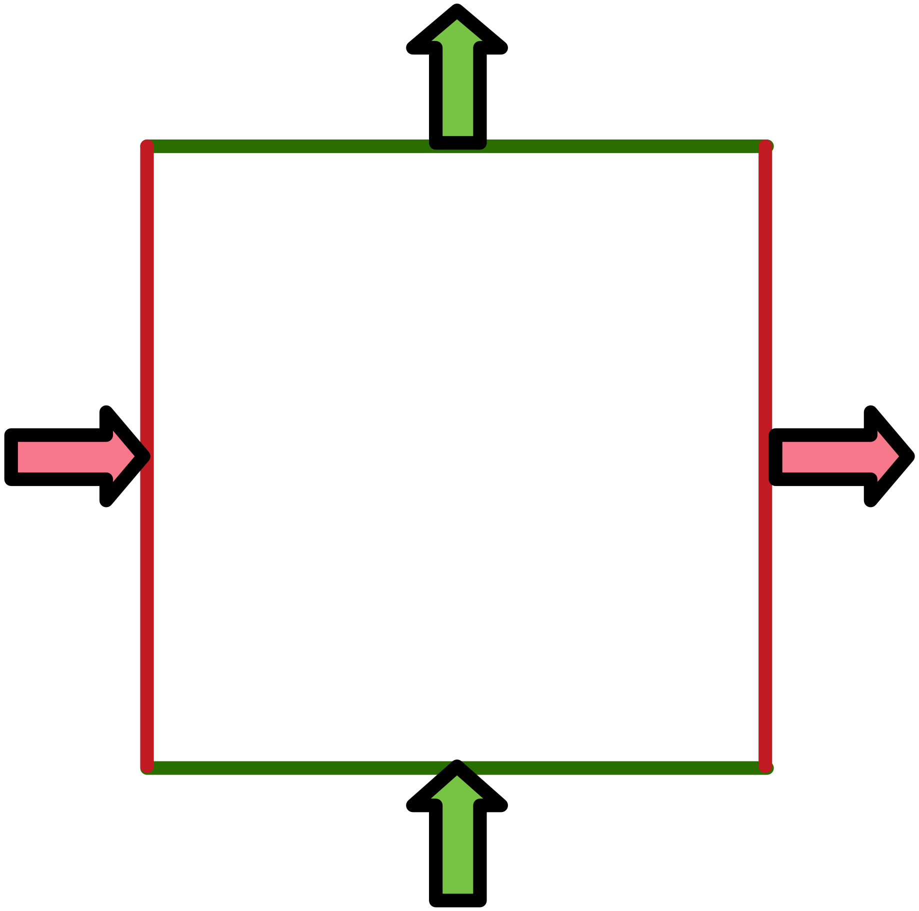
Imagine this room:

If you go out the back wall, you come in through the front.

If you go out the right wall, you come in through the left.

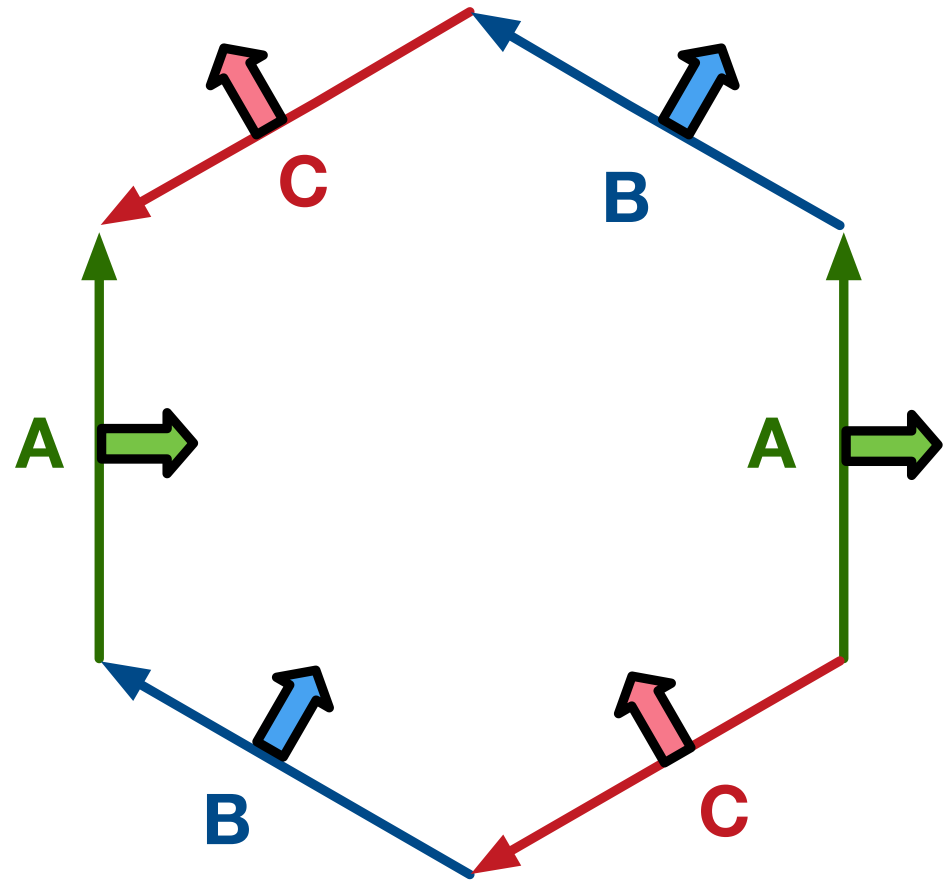


We call this the “flat torus.”

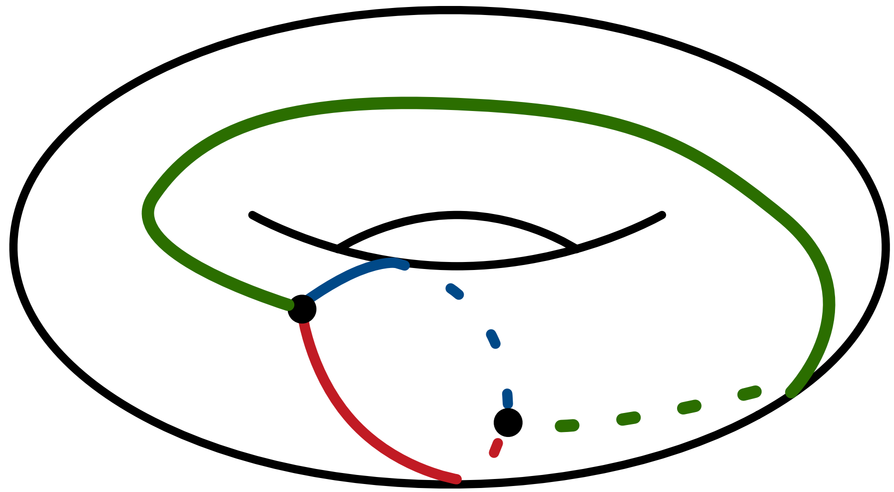


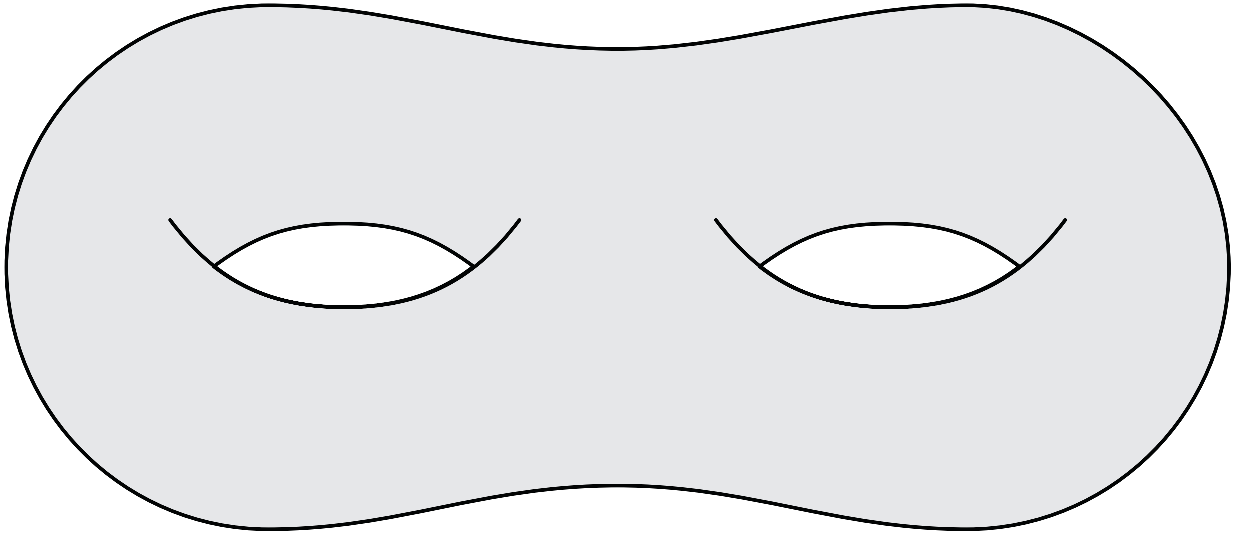
Problem:

Here is another gluing patten with a hexagon. What surface does it create if we glue it back up?



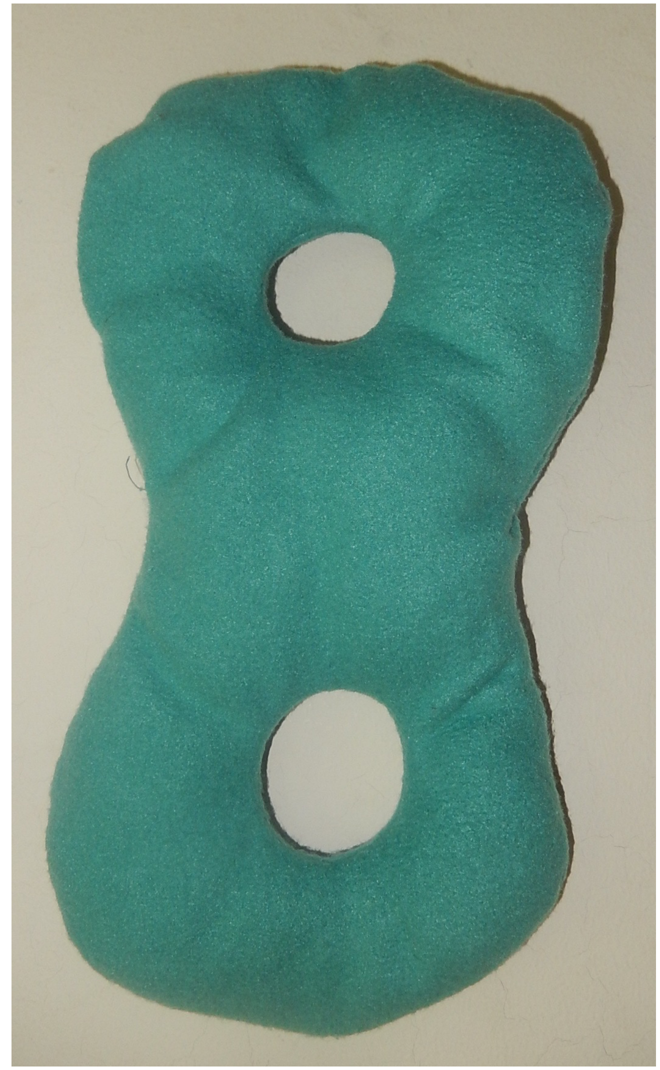
You get the torus again.

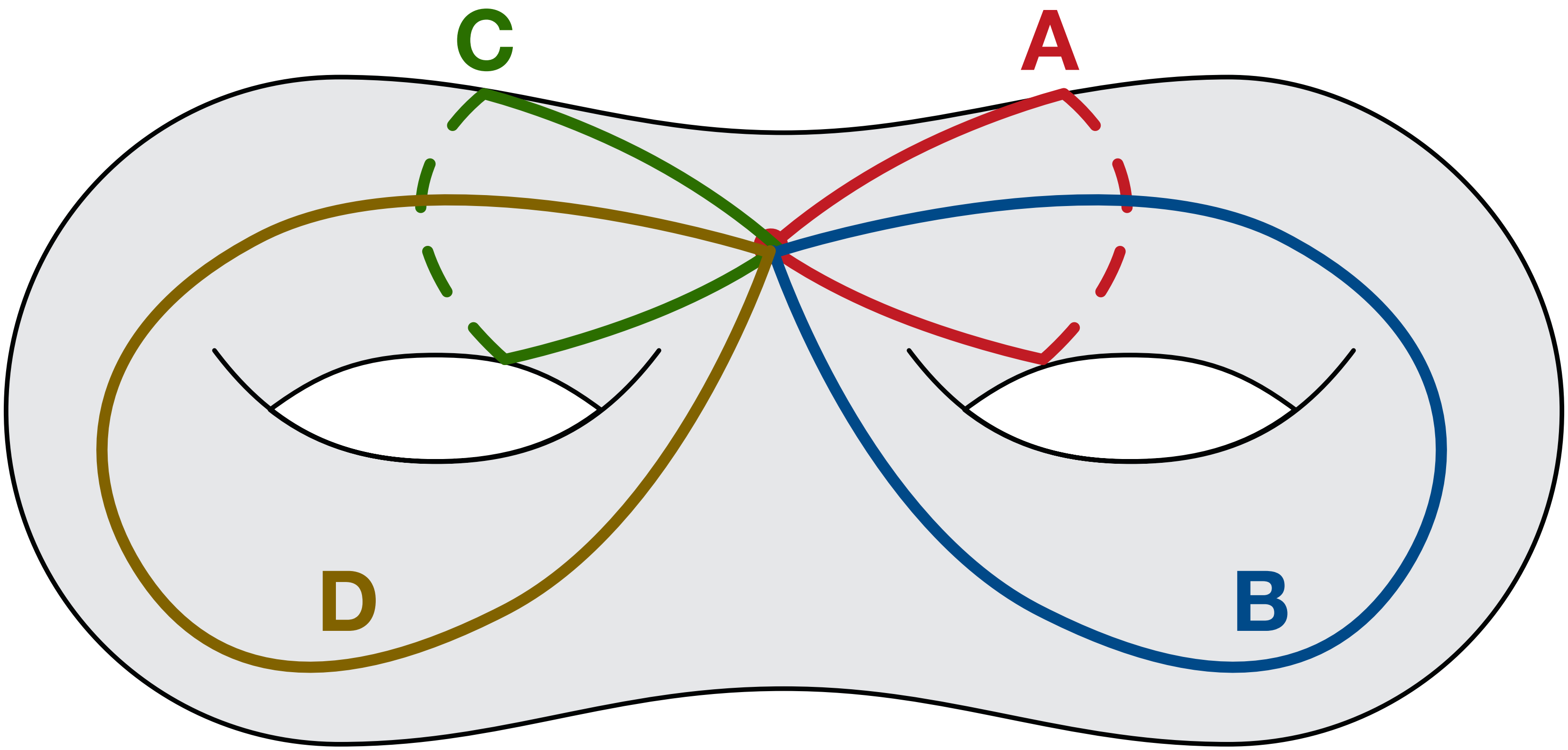




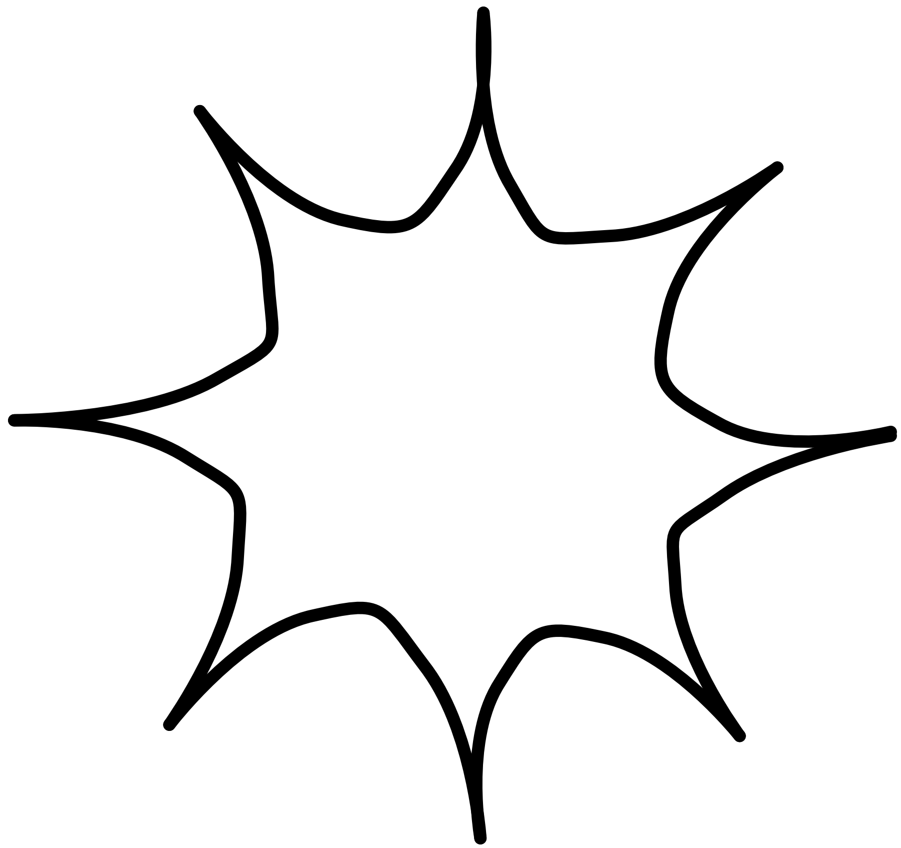
Next, let’s cut up the double torus. There are lots of possible ways to cut it open.

Use the stuffed double torus and yarn to help imagine the cuts. (Please don't actually cut the torus.)

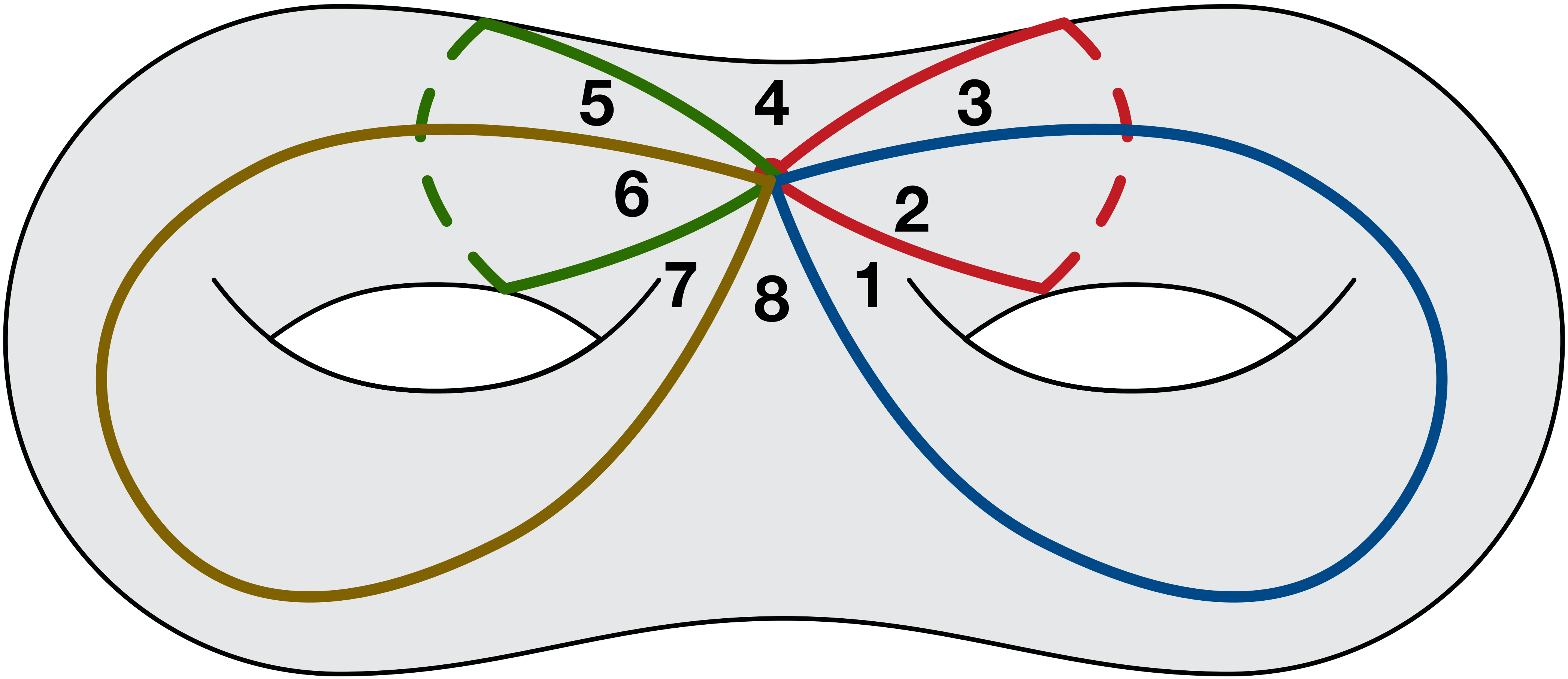




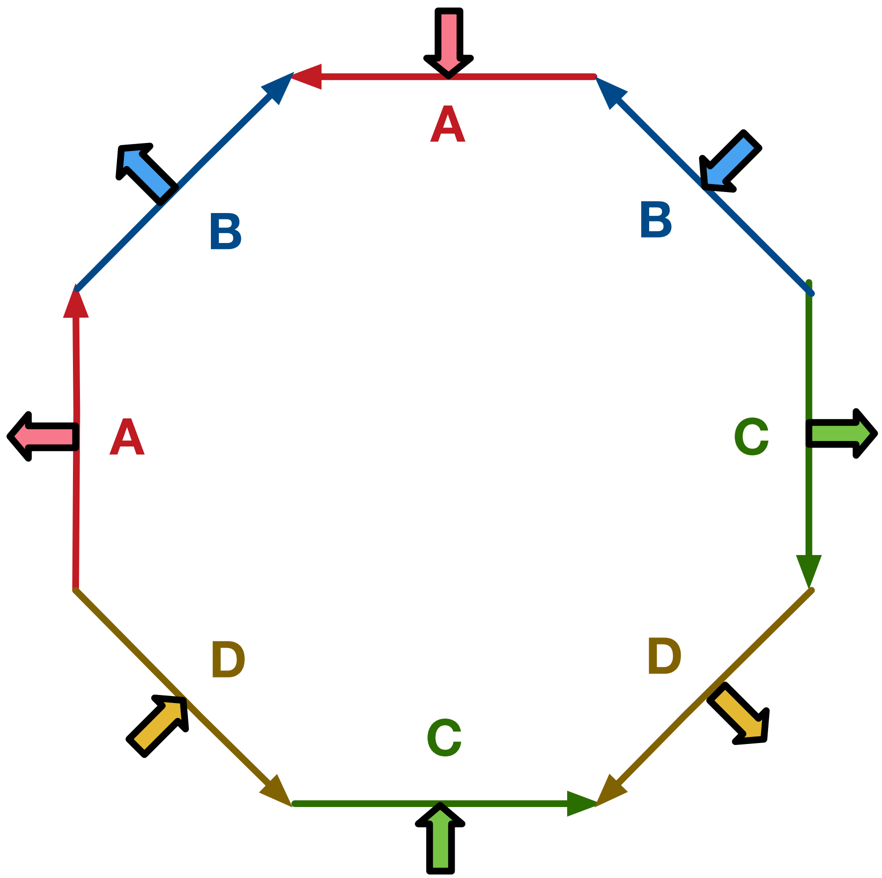
Here is my favorite way to cut open the double torus.



It gives an octagon with pointy corners.



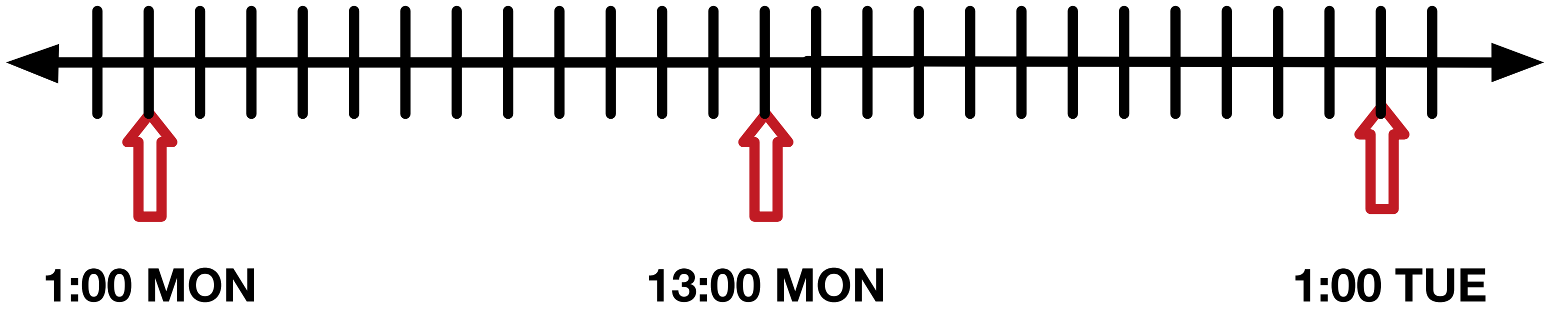
All 8 corners meet at the same point.

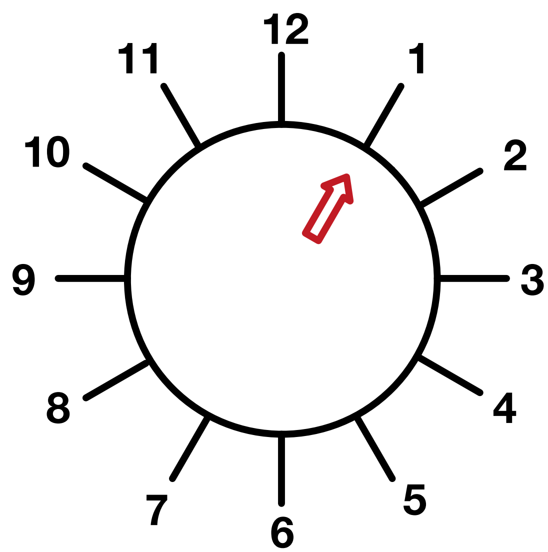


Here is the glueing pattern.

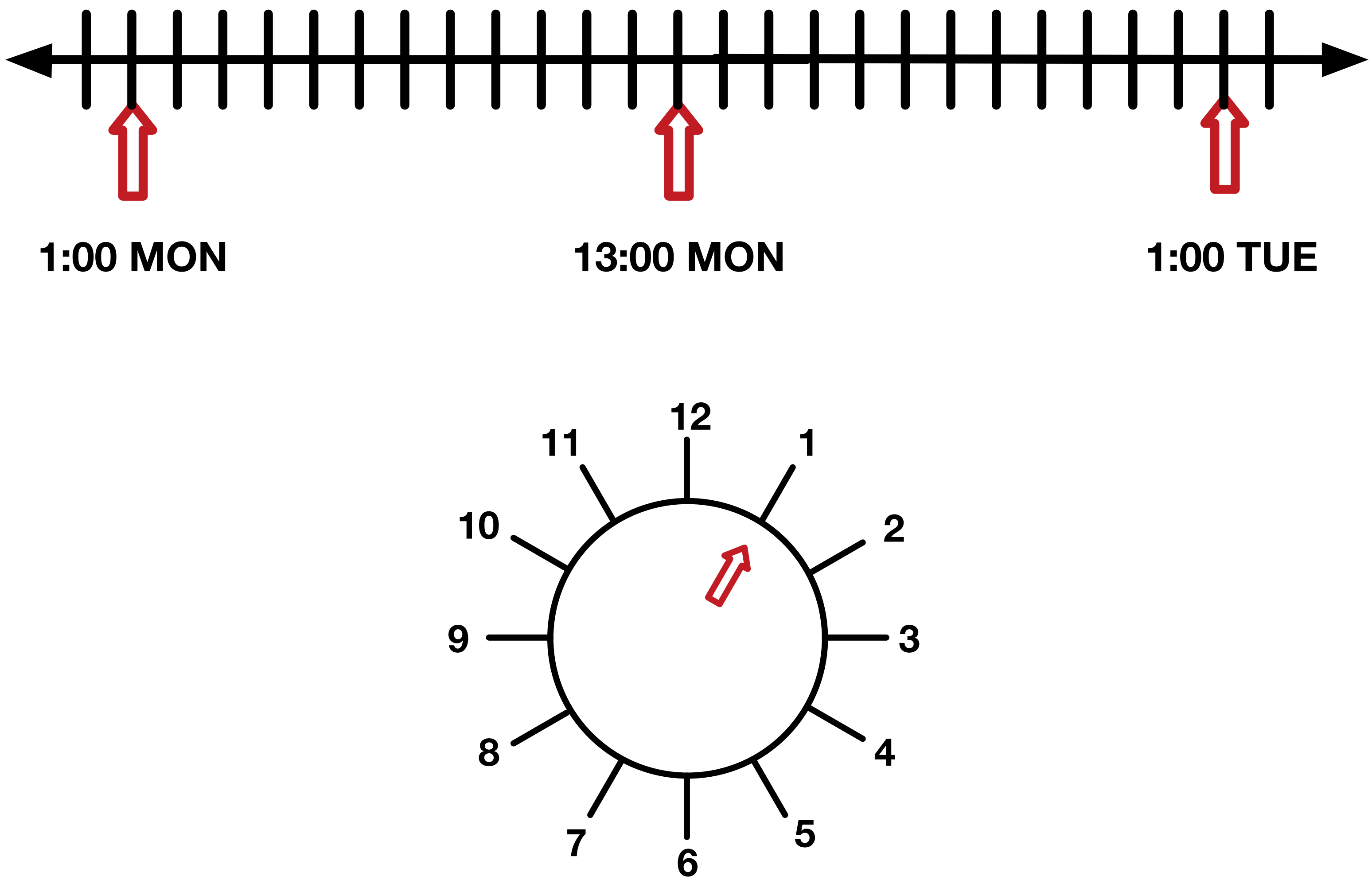
Covering Maps

"Map" from infinite time line
to clock.



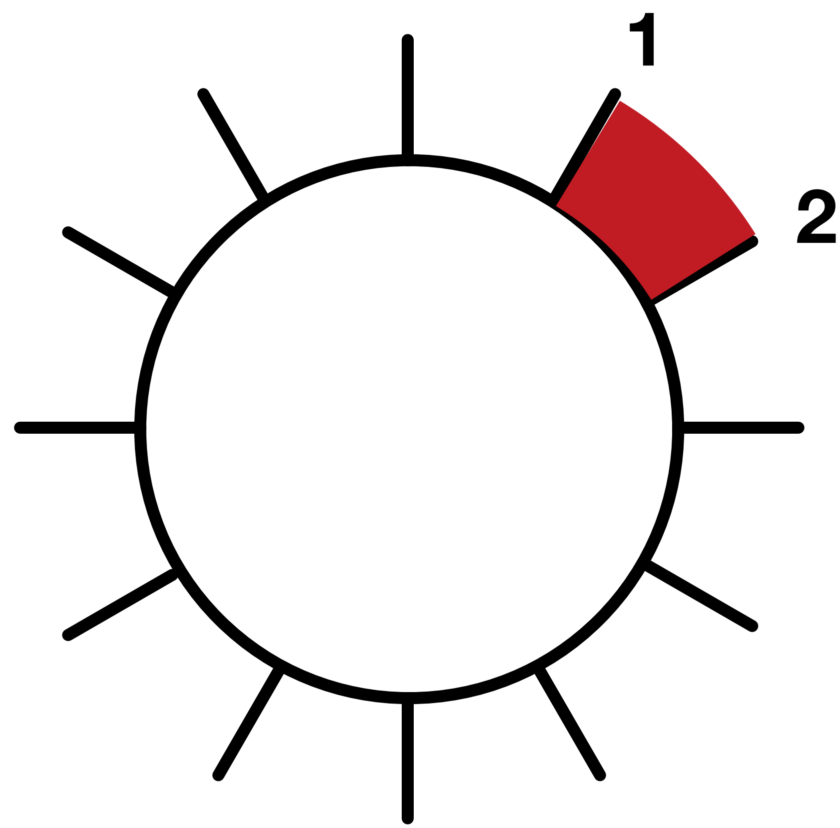


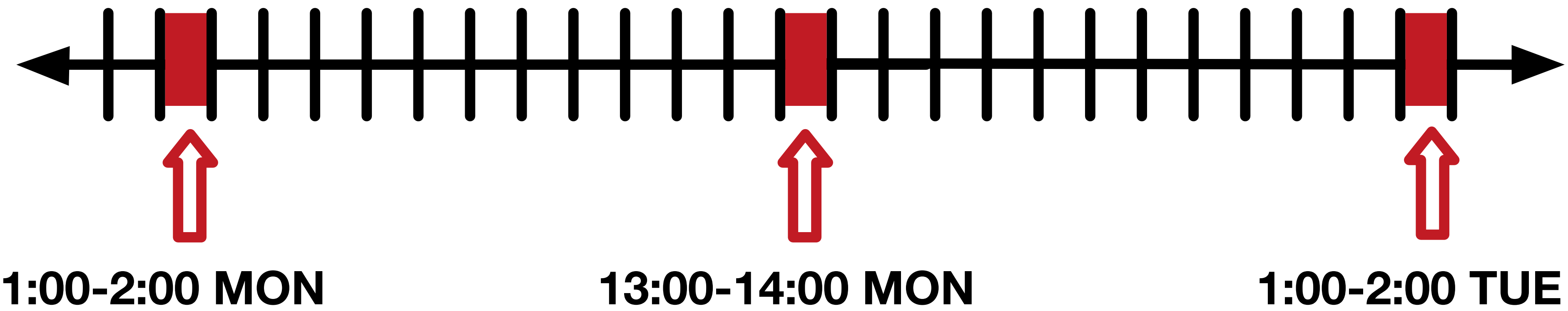
Every 1am and 1pm map to the same "1" on clock.

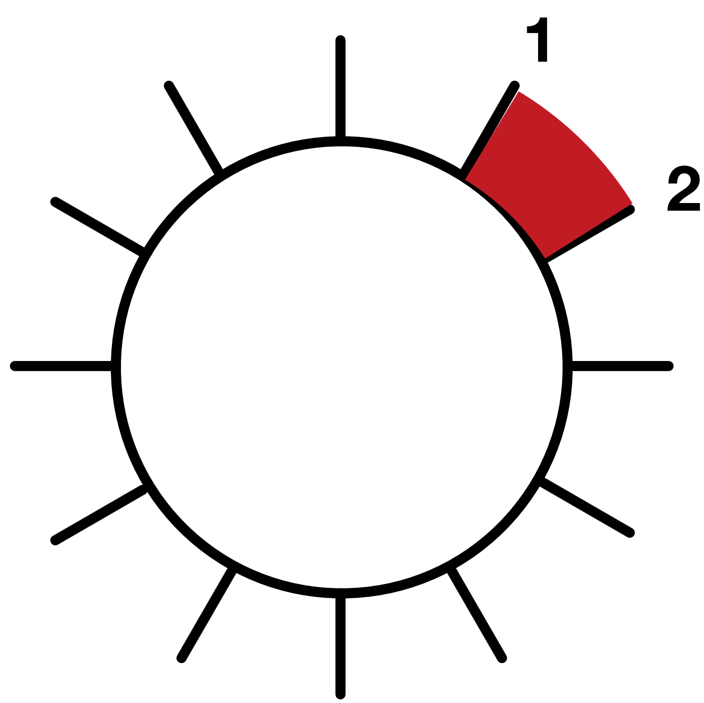


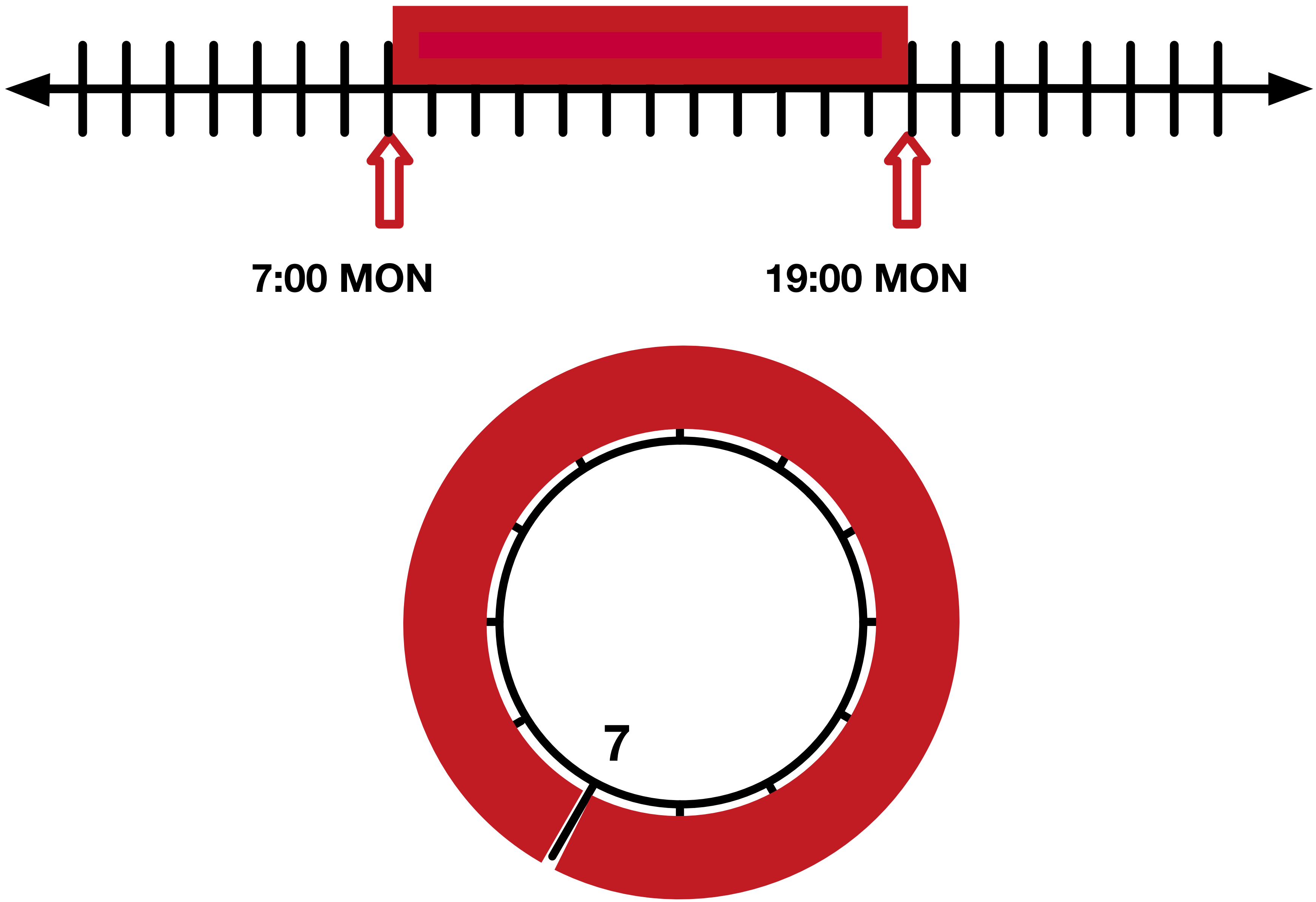
We call this function an "infinite covering map"

What times on the time line map to the interval (1,2) on the clock?



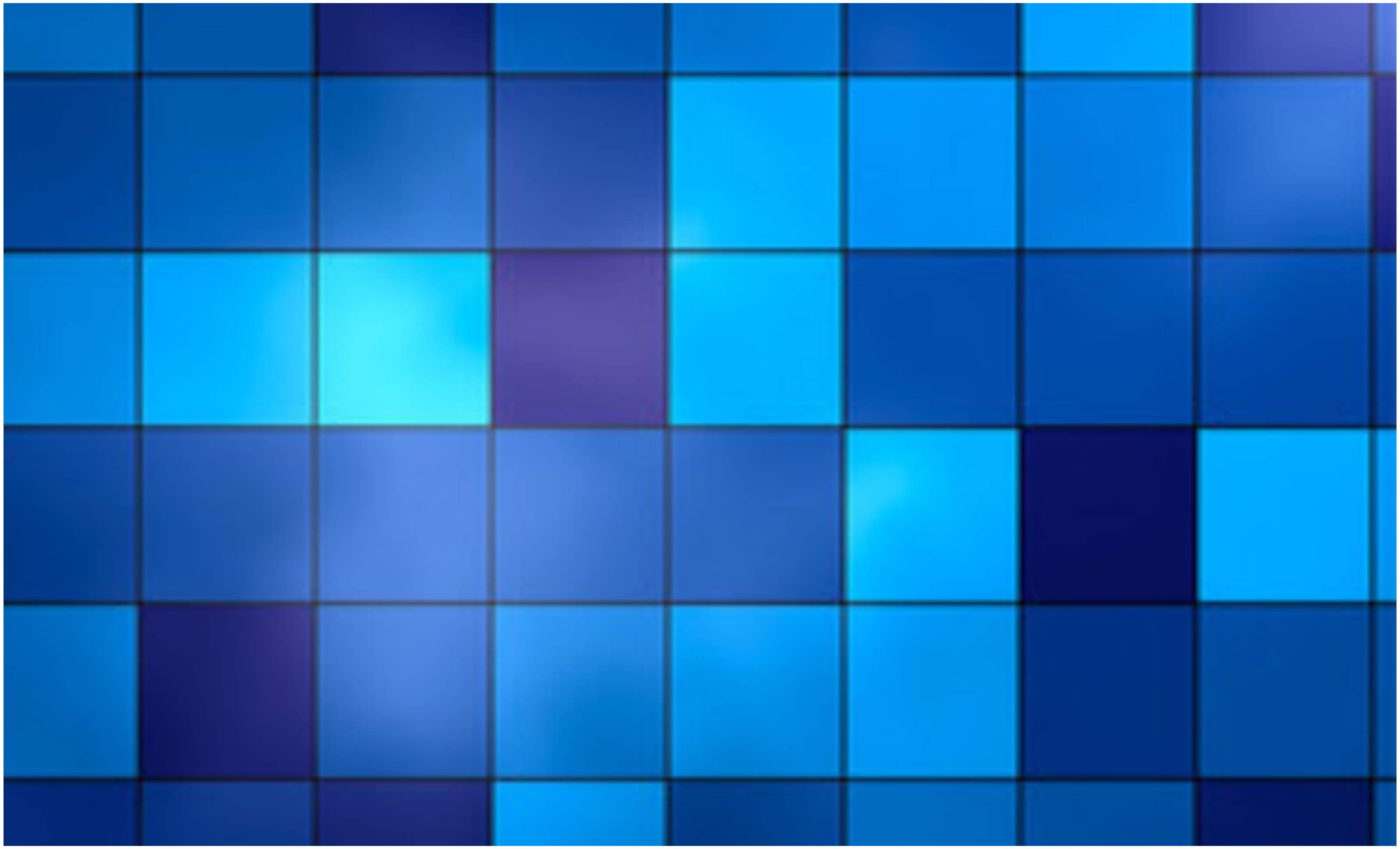

1:00-2:00 MON
13:00-14:00 MON
1:00-2:00 TUE


1
2

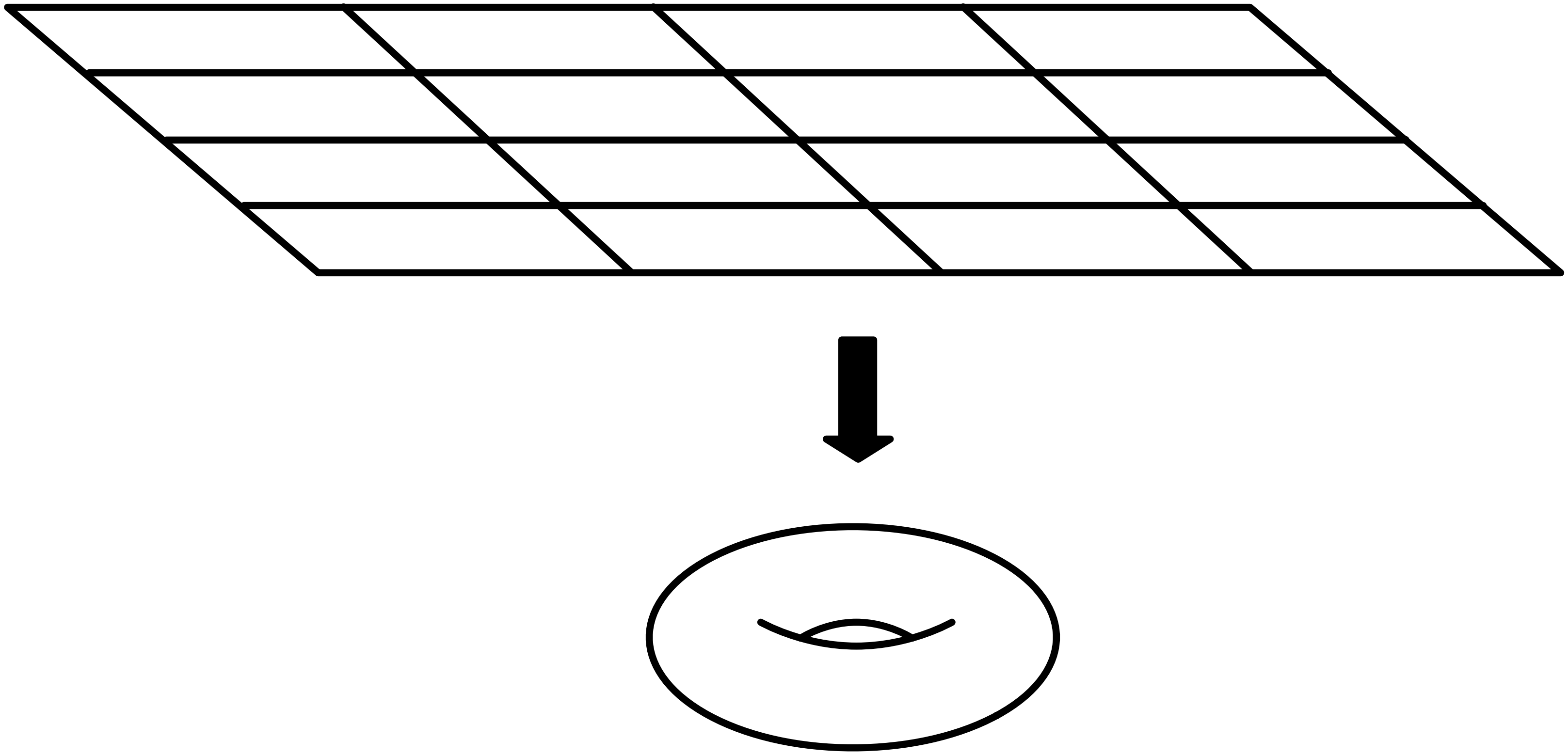


A half day covers the whole circle.

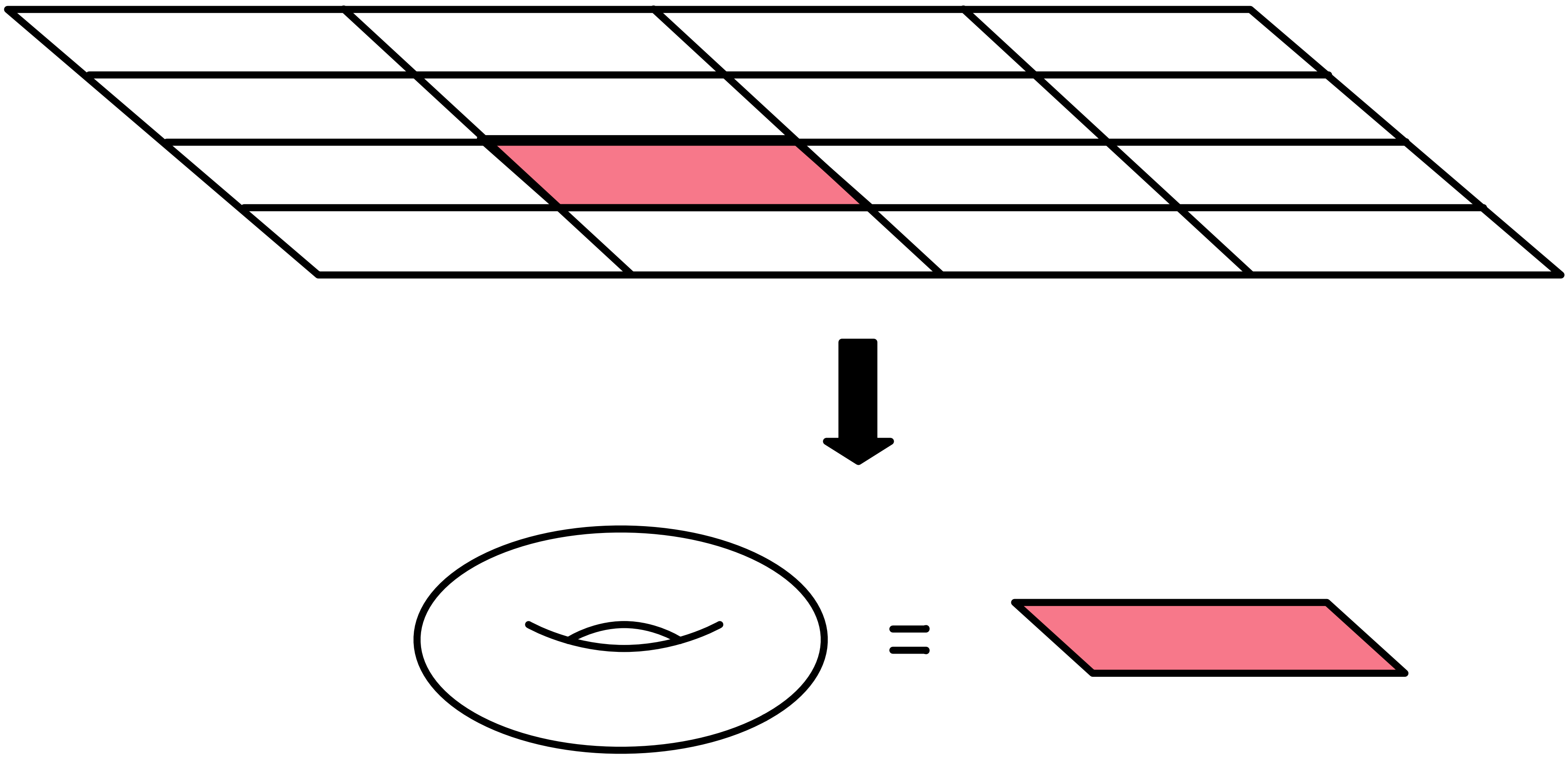
Covering maps in two dimensions



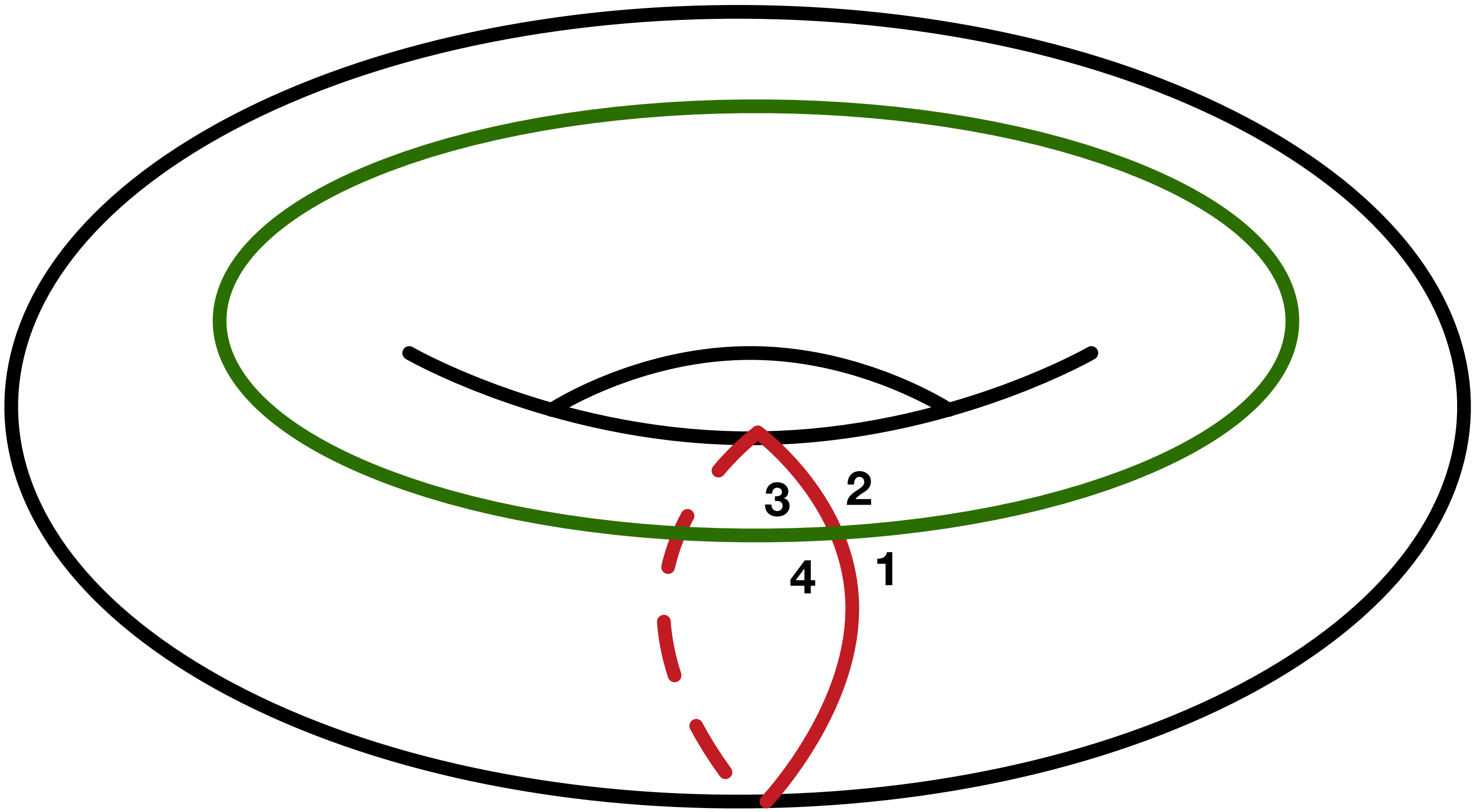
From last time: the (4,4) tiling of Euclidean plane



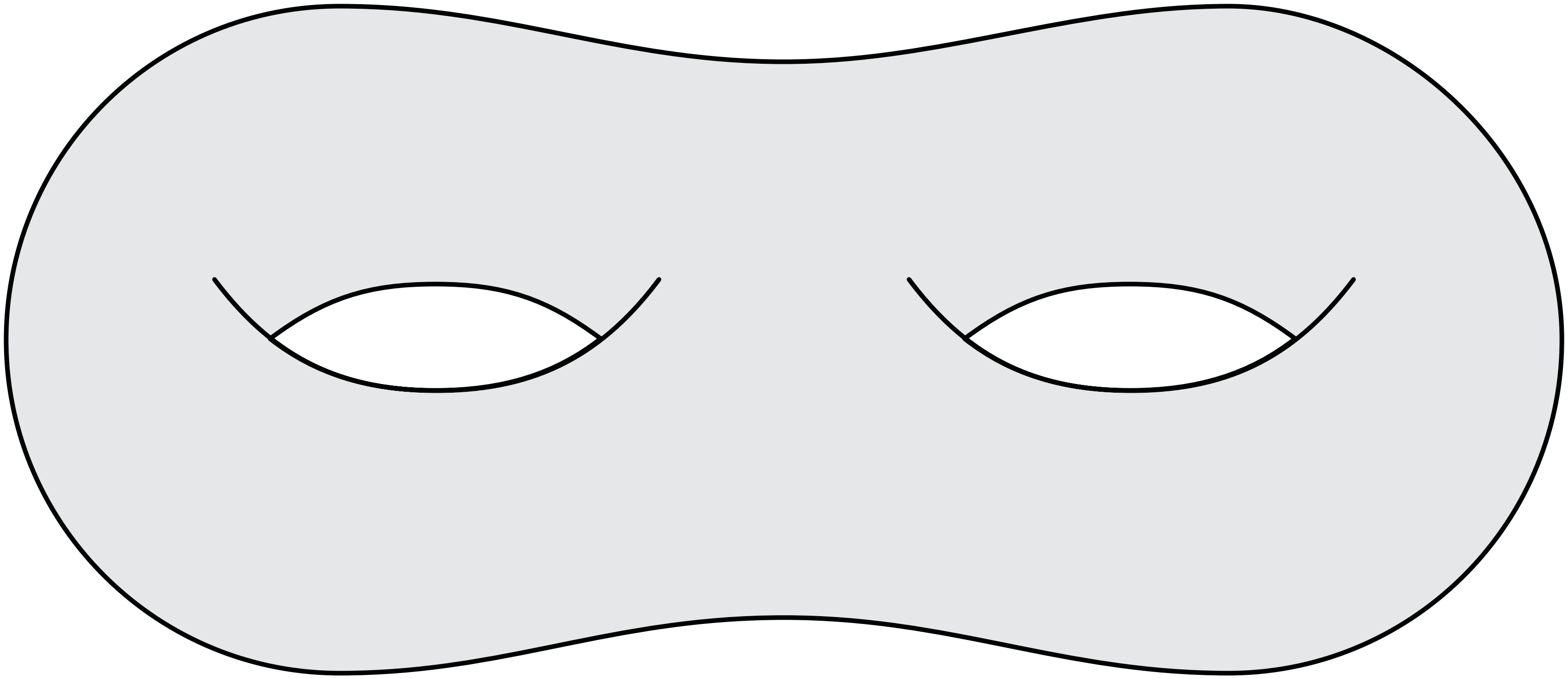
We can cover the torus
with the Euclidean plane.



Each square in the tiling
covers the torus once.

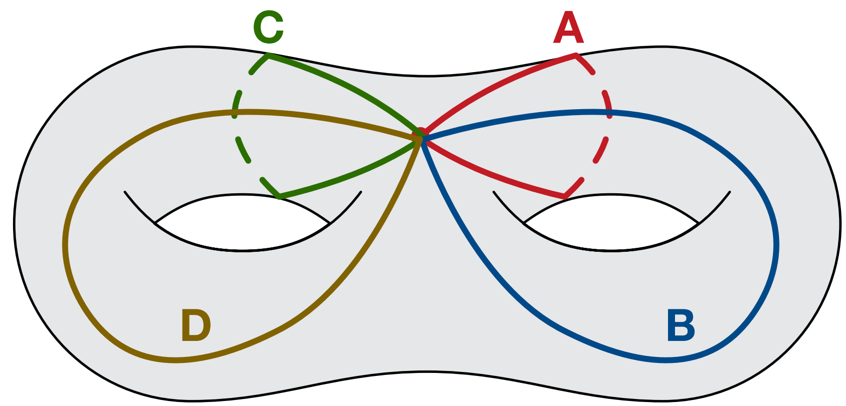


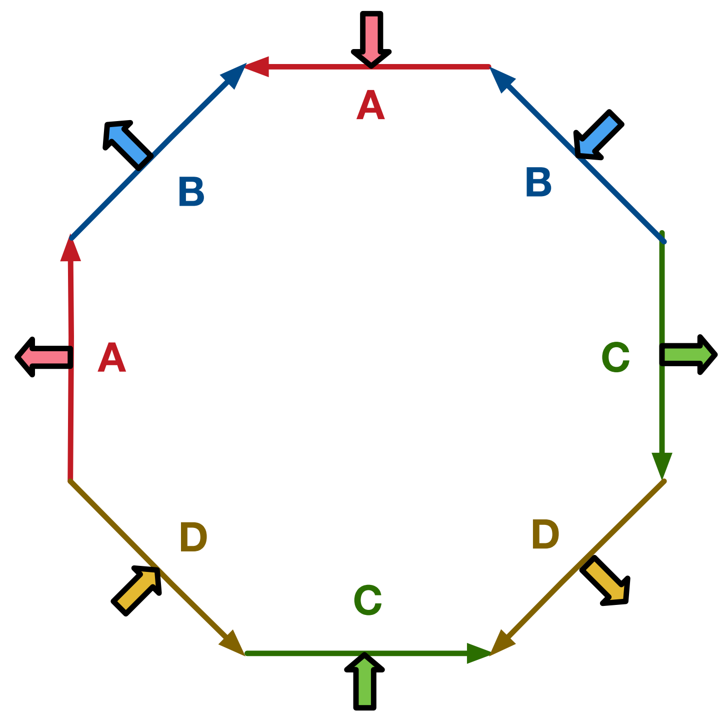
When we glue, four corners meet at one point, just like in the tiling.



Double Torus

We can unwrap the double torus into an octogon.

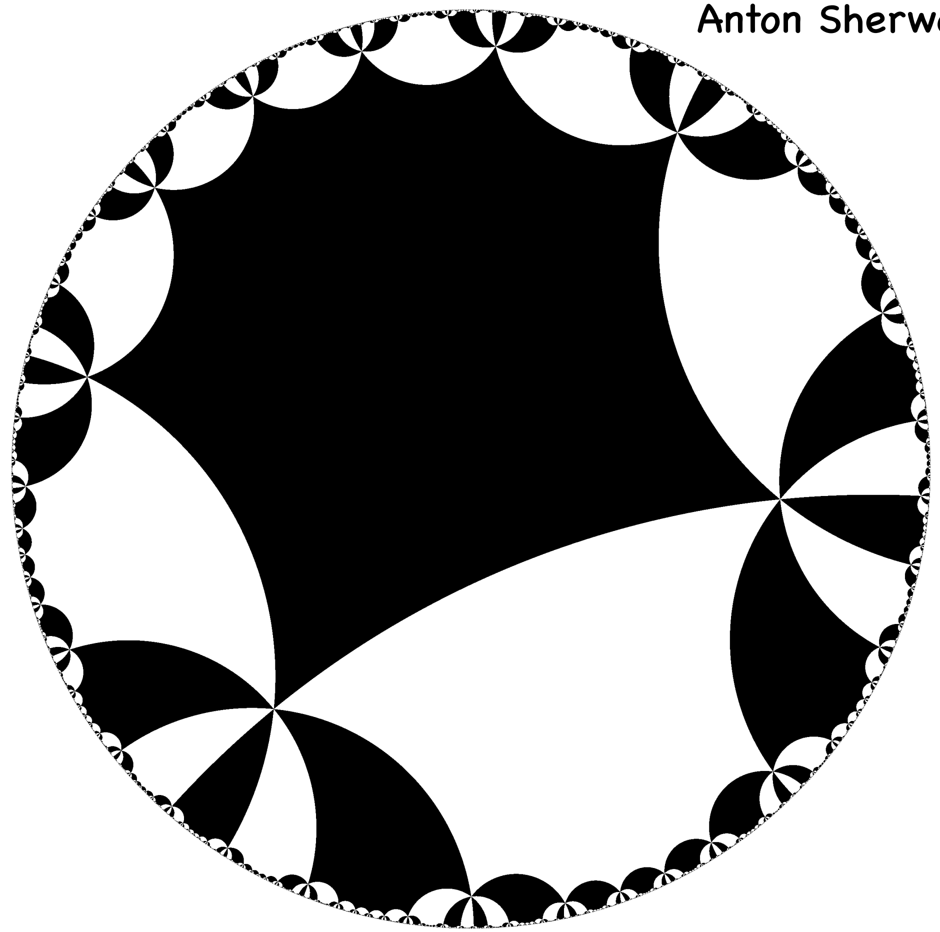


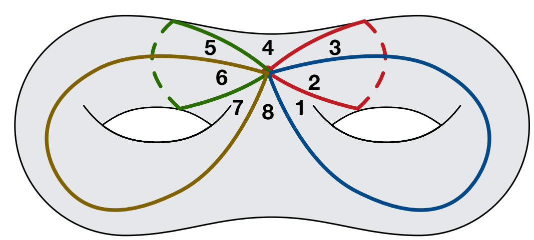


What space can we tile with octagons, meeting 8 to a vertex?

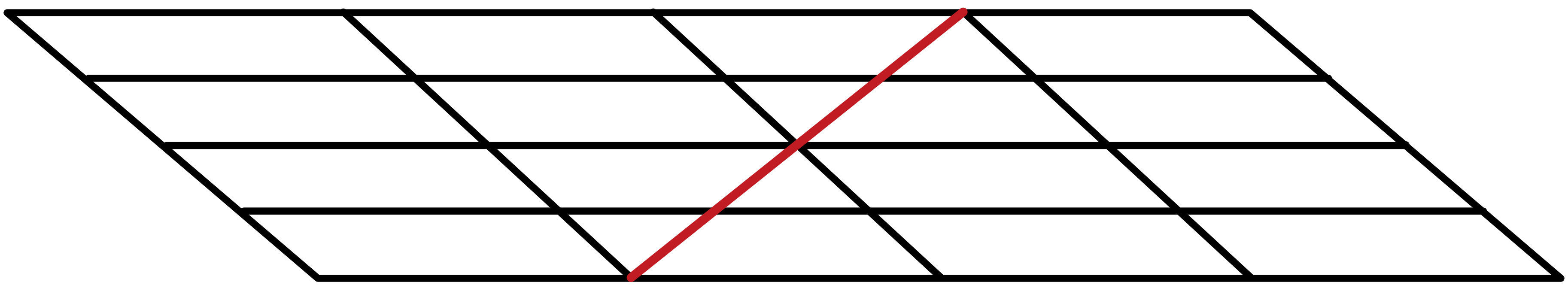
We can use an (8,8) tiling of hyperbolic space to cover the double torus.

Picture:
Anton Sherwood

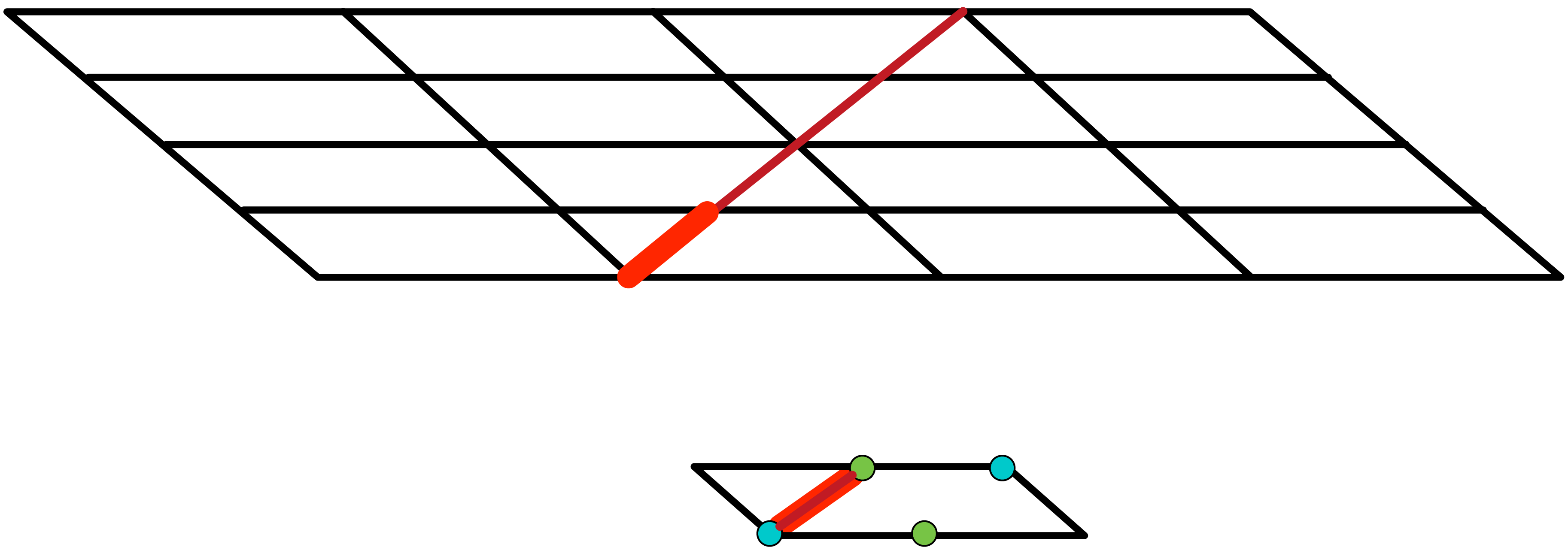




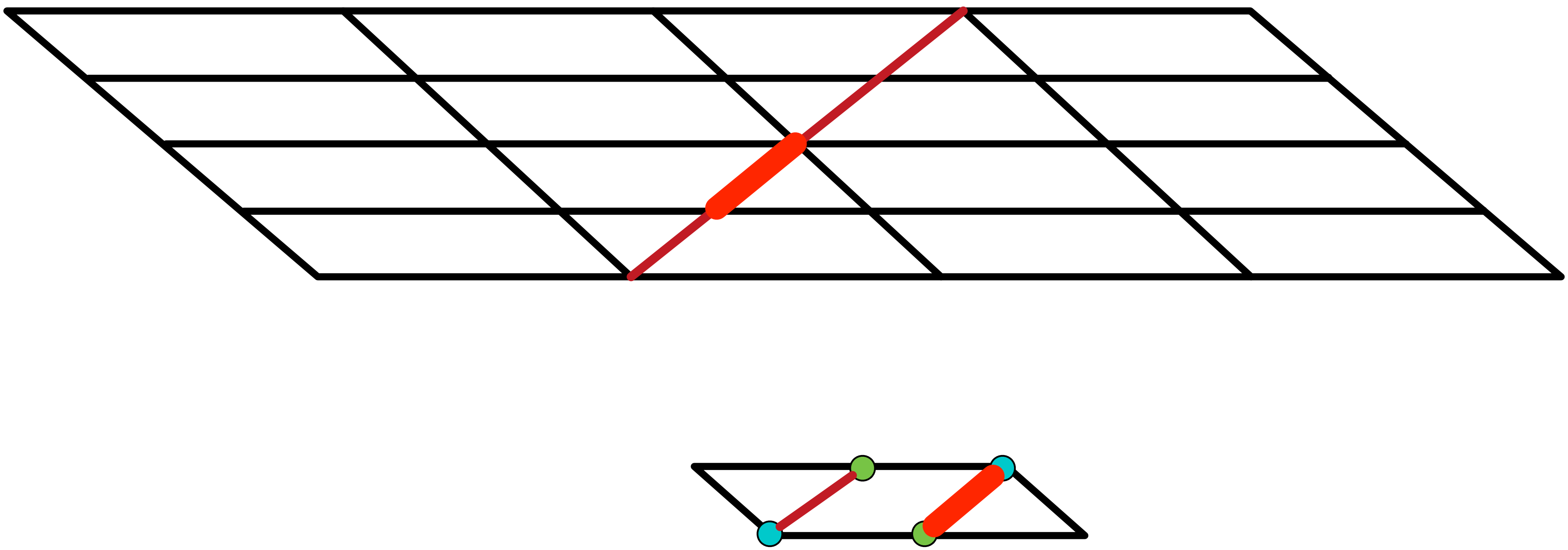
These covering spaces give us one way to define “straight lines” on the surfaces.



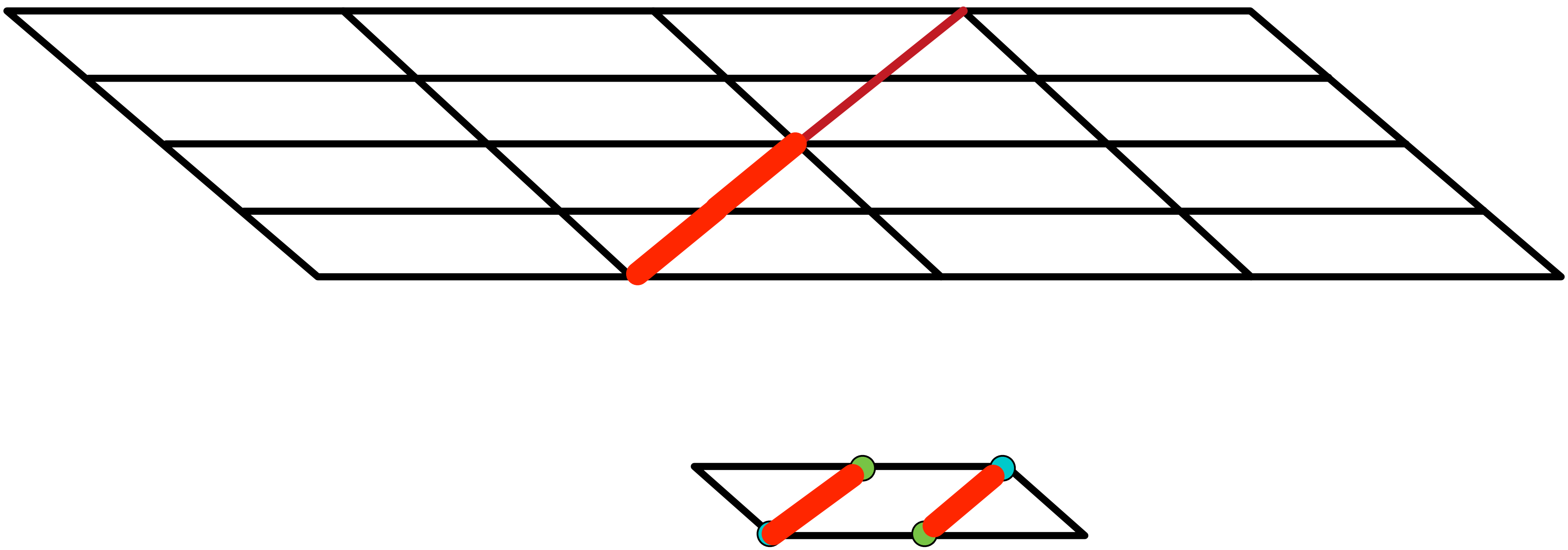
Here is a line on the plane. How does it map down to the torus?



The first segment of the line maps down to a segment in the torus.

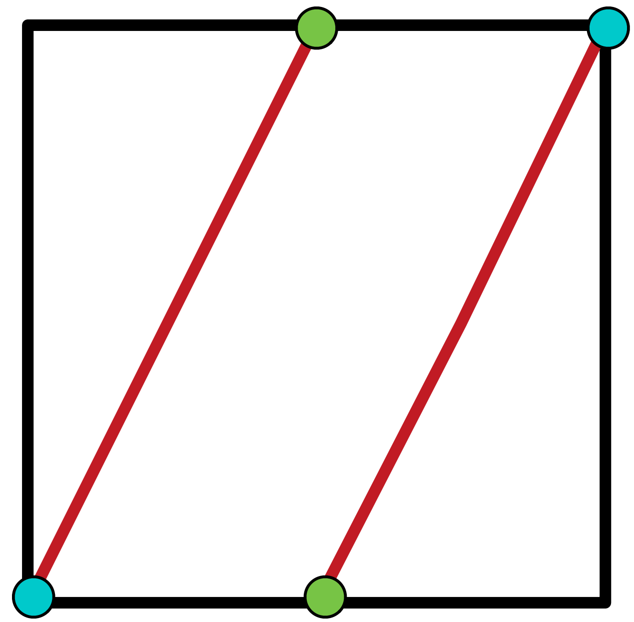


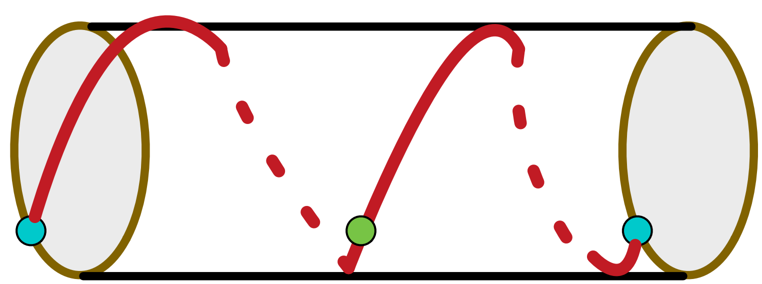
The second segment of the line also maps down to a segment in the torus.



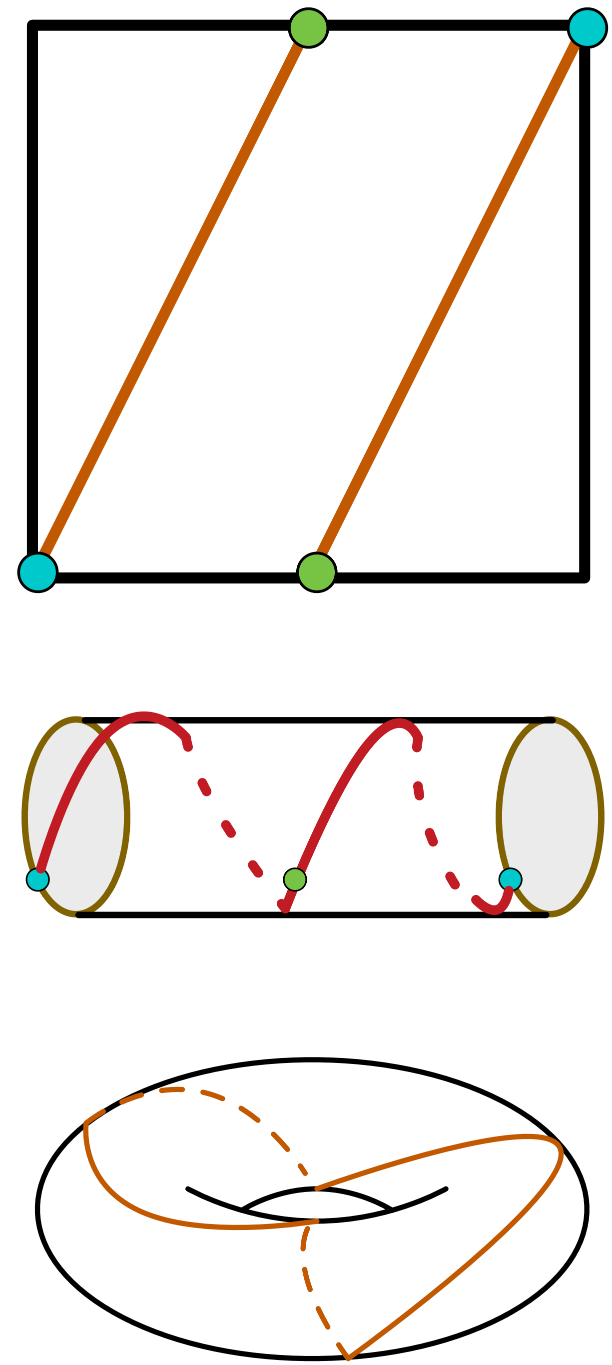
The line traces over and over these two segments.

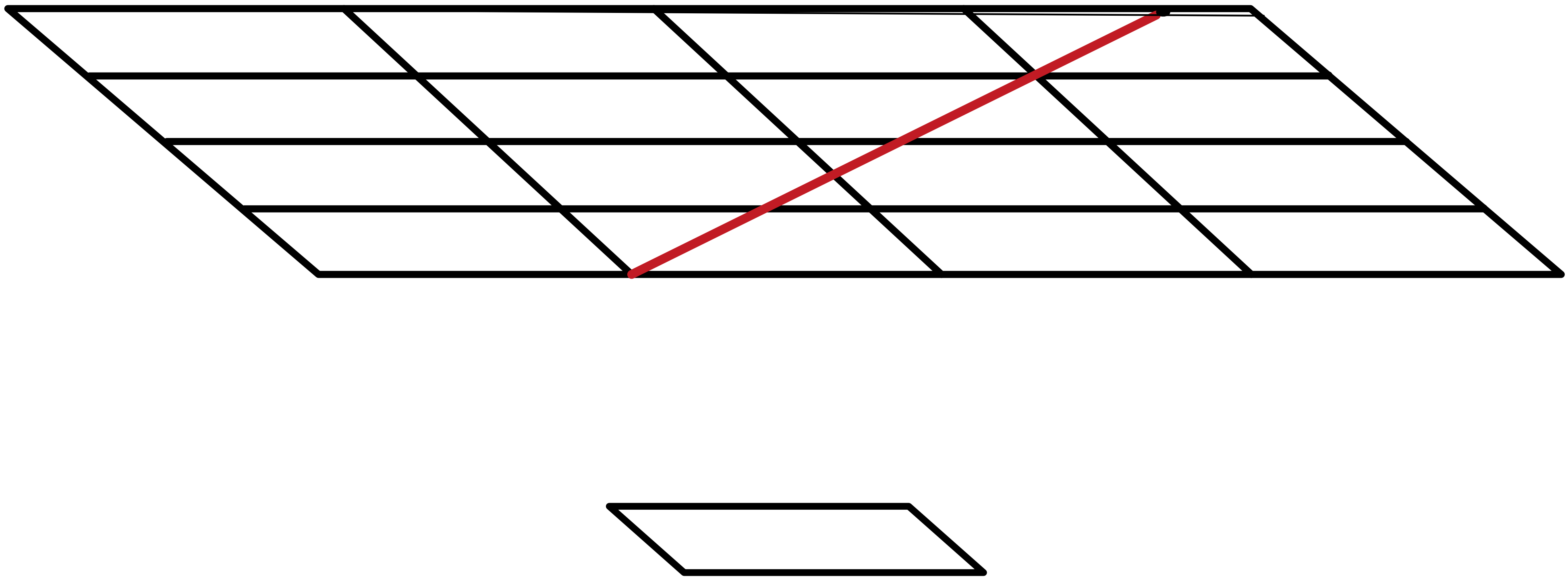
We can glue the square back into a cylinder.





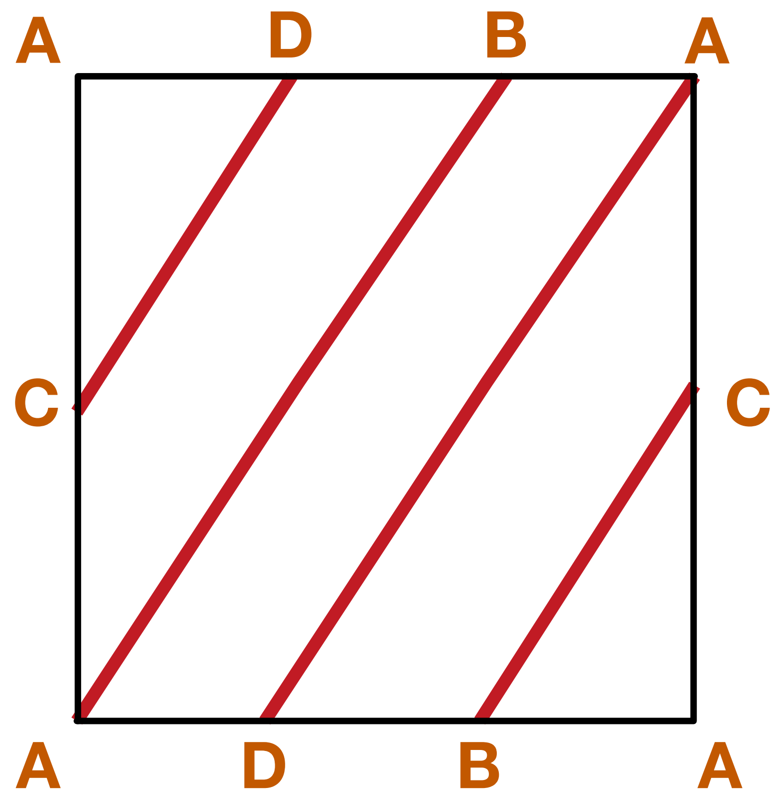
And then glue the cylinder back into a torus, so the two segments make a curve.

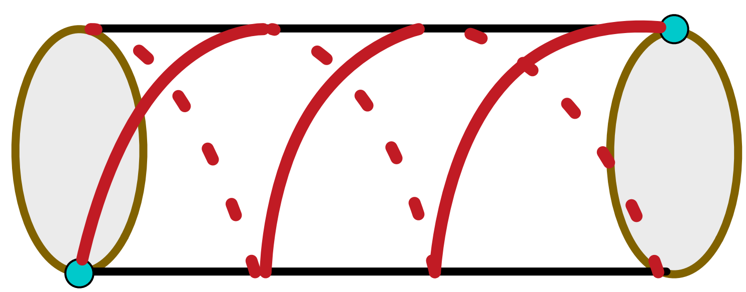




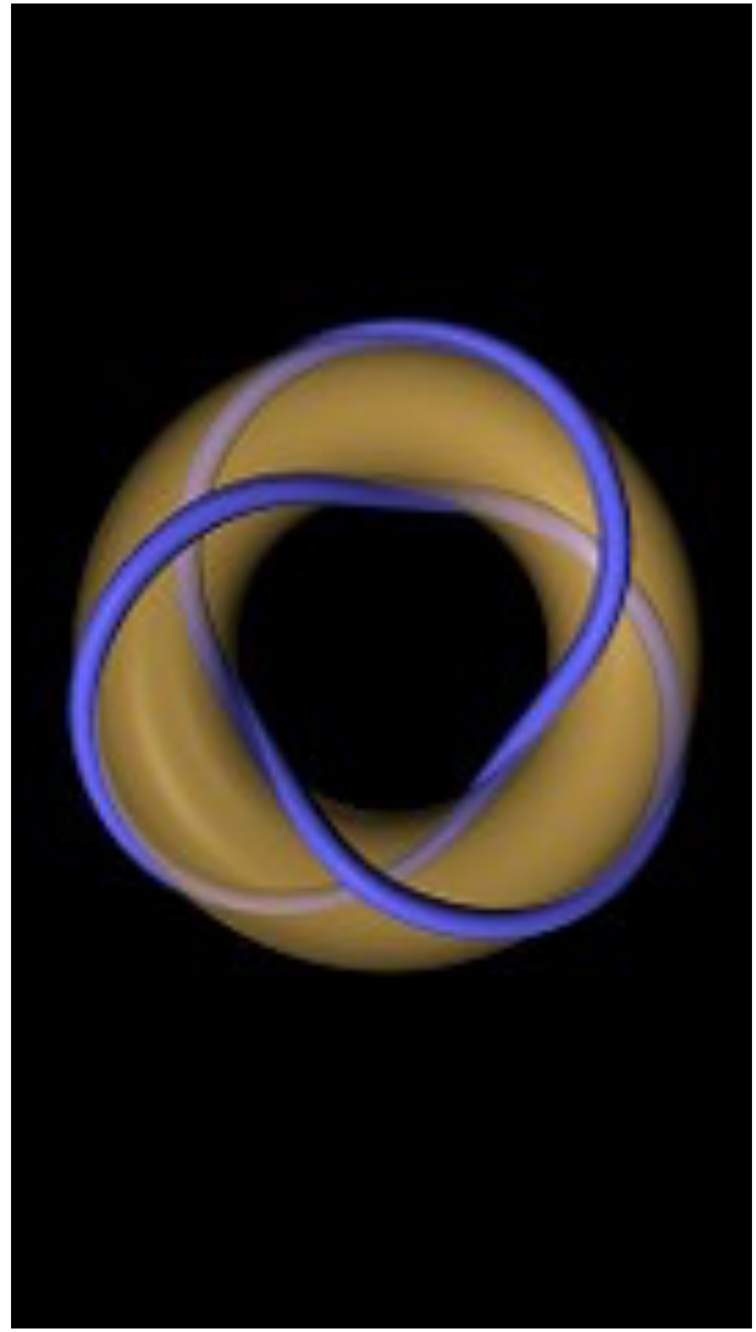
How would a line of slope 3/2 map down to the torus?

Here is the line of slope 3/2 on the torus.

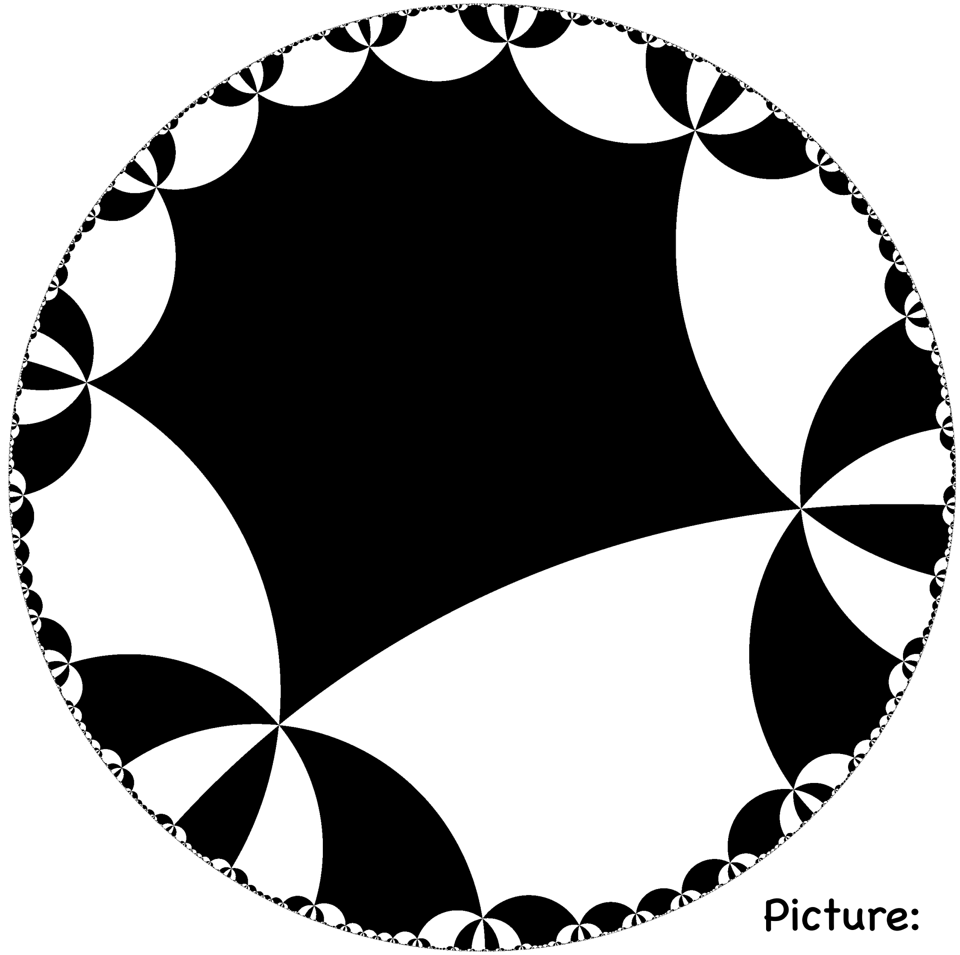




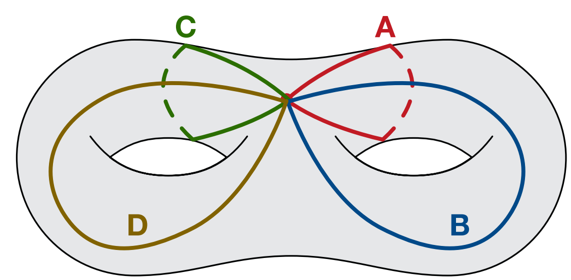
A line of slope 3/2 goes 3 times through the hole and 2 times around it.



We can also put "hyperbolic straight lines" on the double torus.

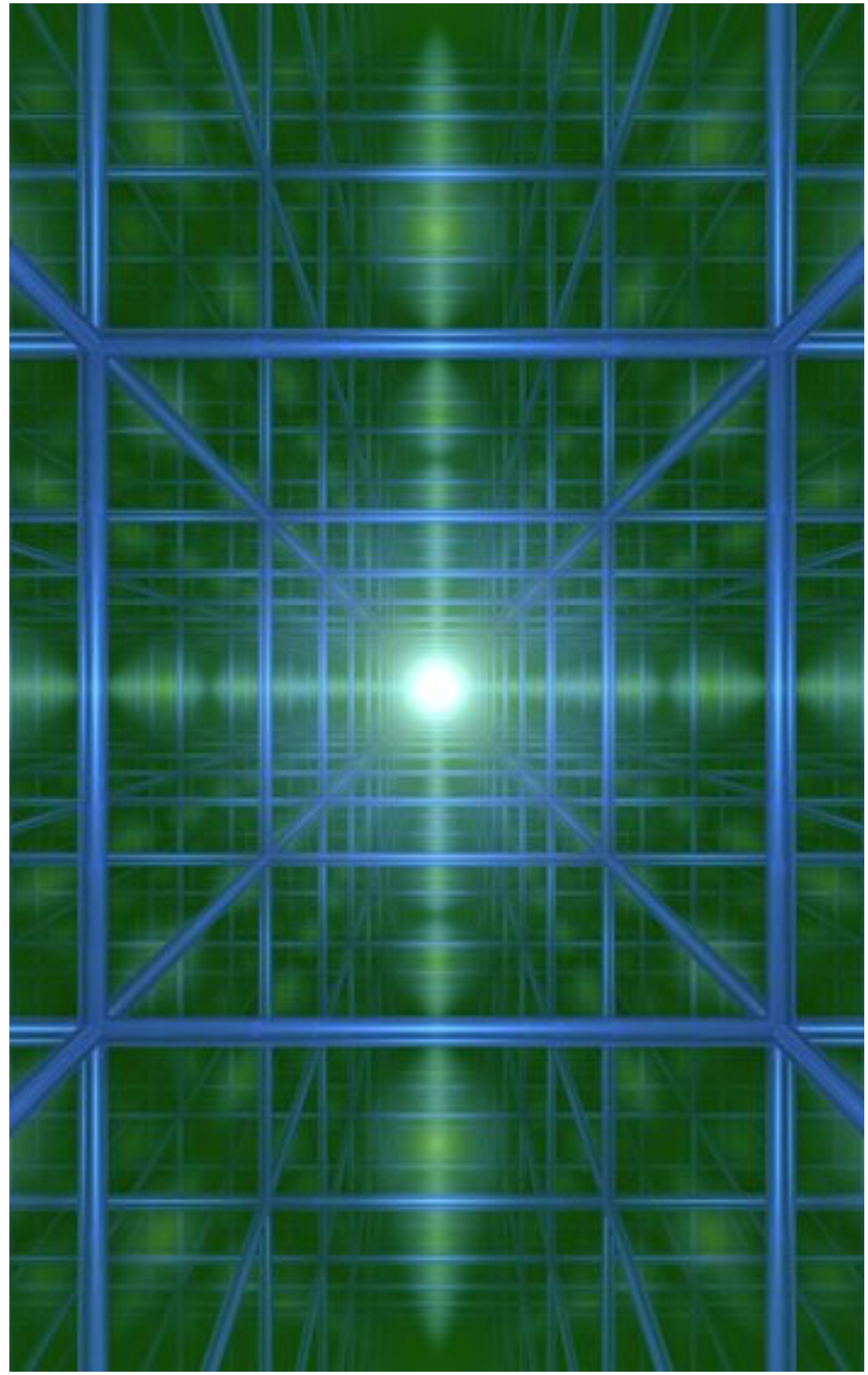


Picture: Anton Sherwood

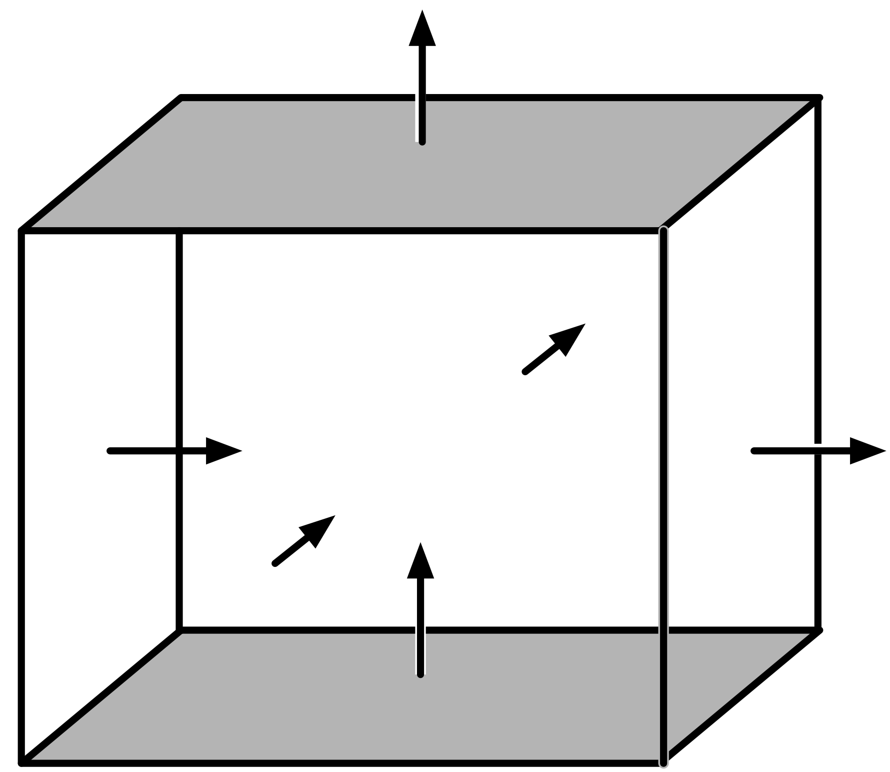


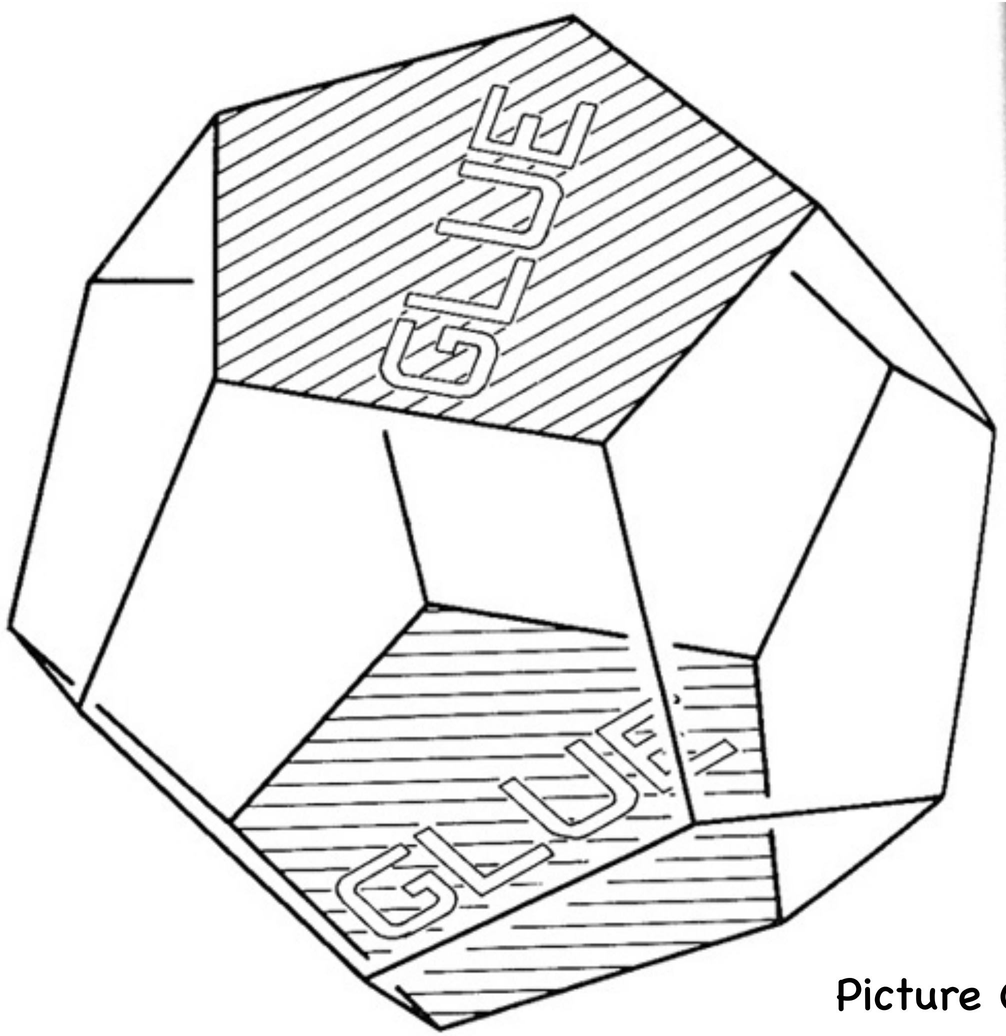
Higher dimensions

This Euclidean tiling by cubes covers the 3-dimensional torus.



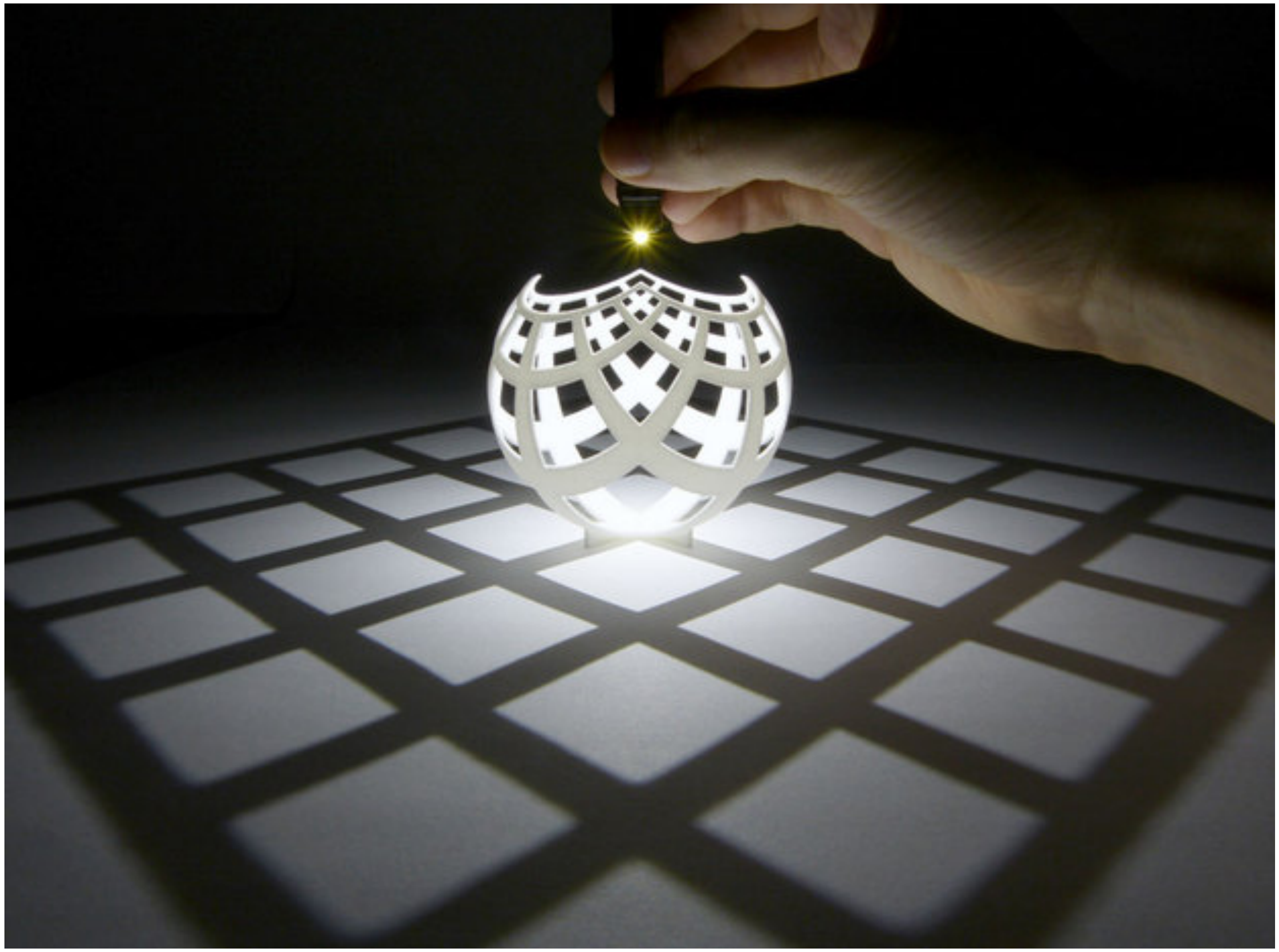
In a three dimensional torus, if you go out the ceiling, you come back up through the floor.





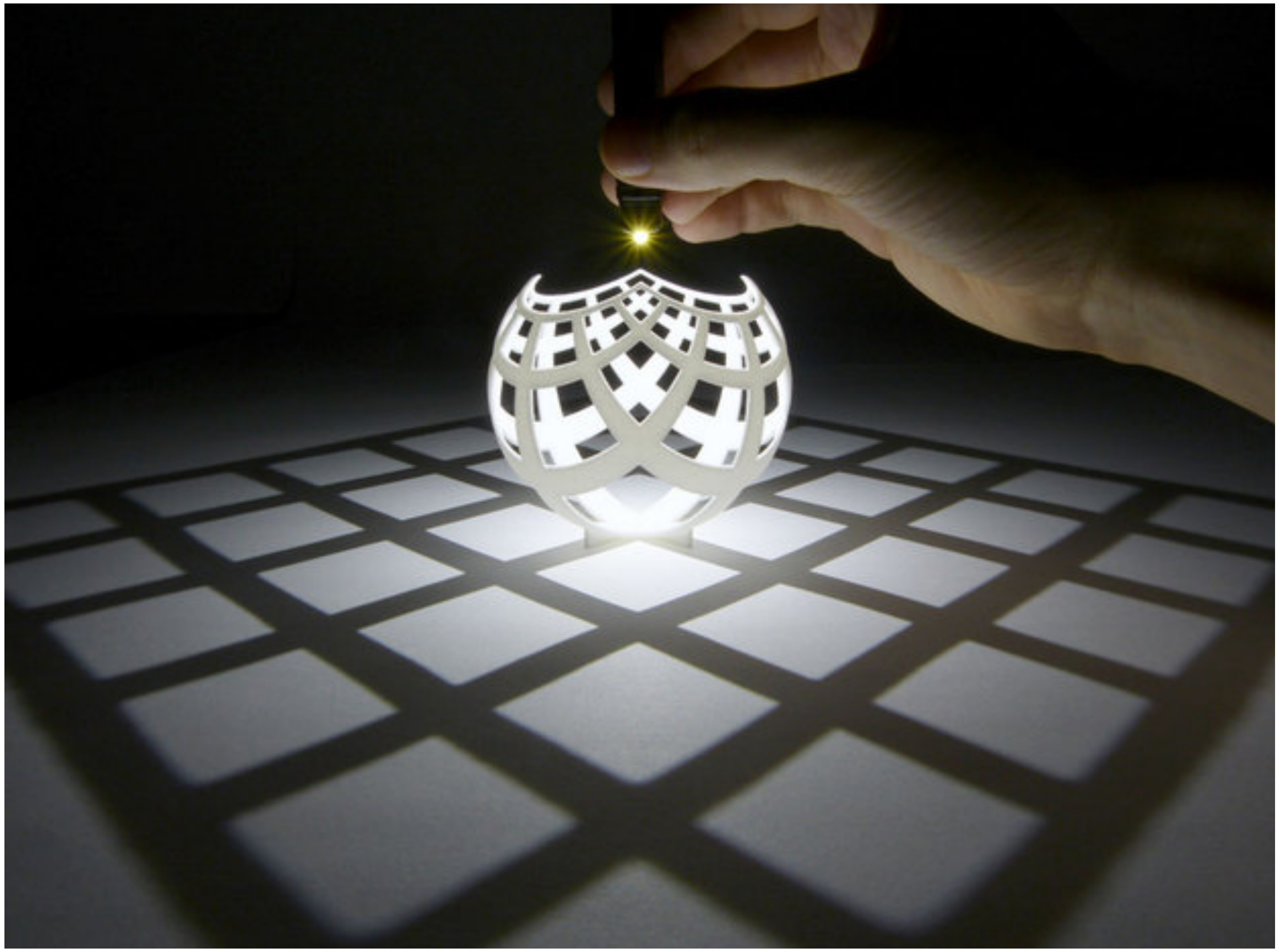
Picture Credit: Weeks 1985

Poincaré Dodecahedral Space: Glue opposite faces with a 1/10 turn. This space is covered by the 3-sphere.



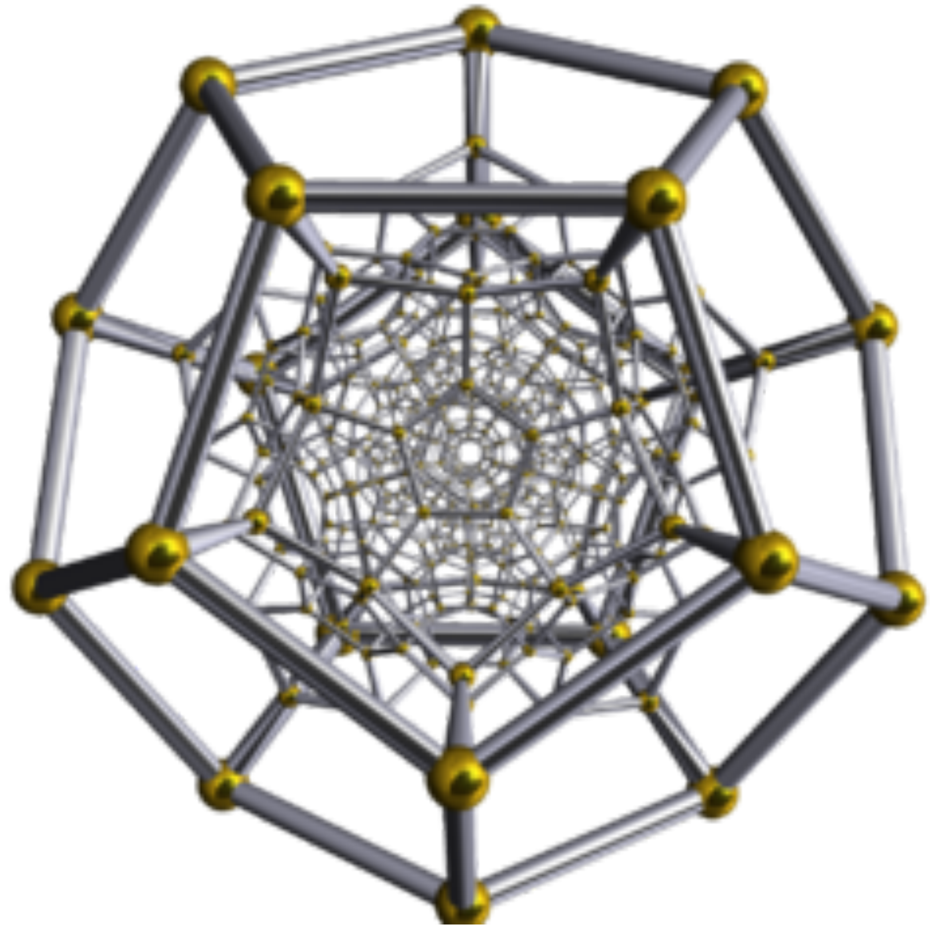
sculpture by Henry Segerman

To make the 2-sphere, add one point "at infinity" to the plane.



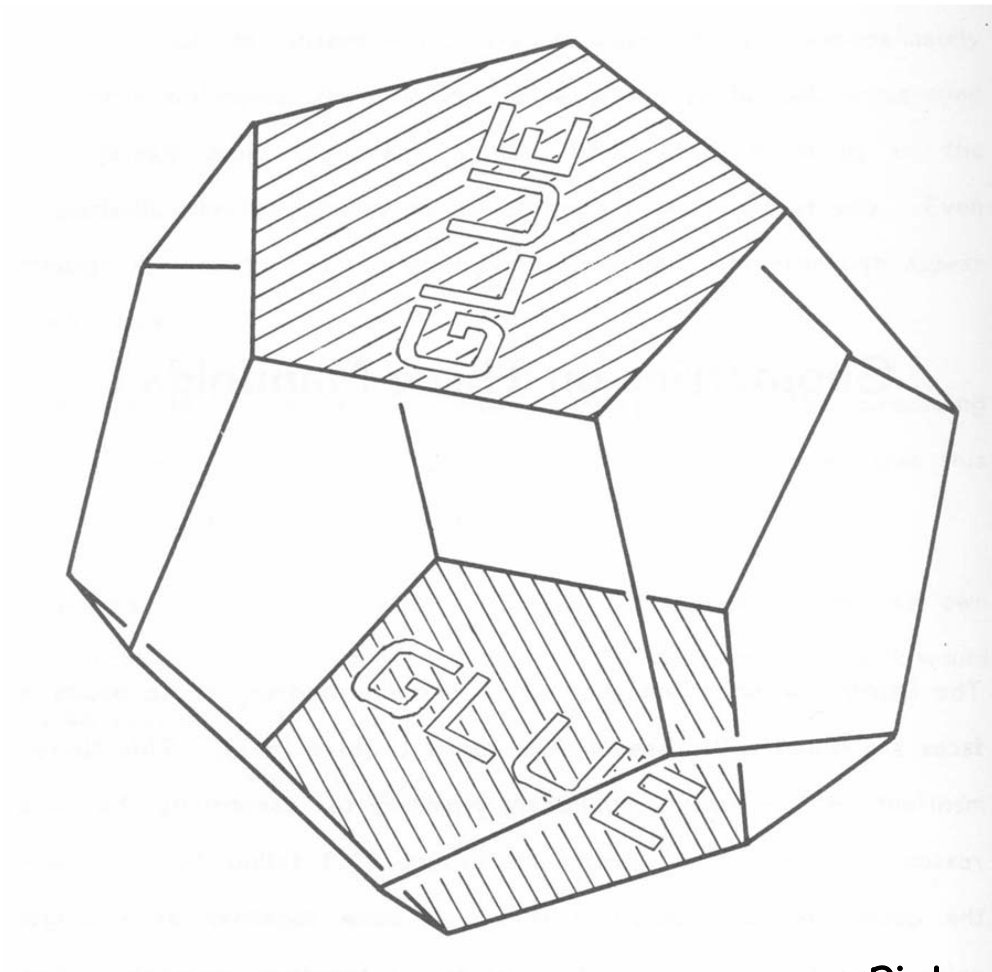
sculpture by Henry Segerman

To make the 3-sphere, add one point “at infinity” to three space.



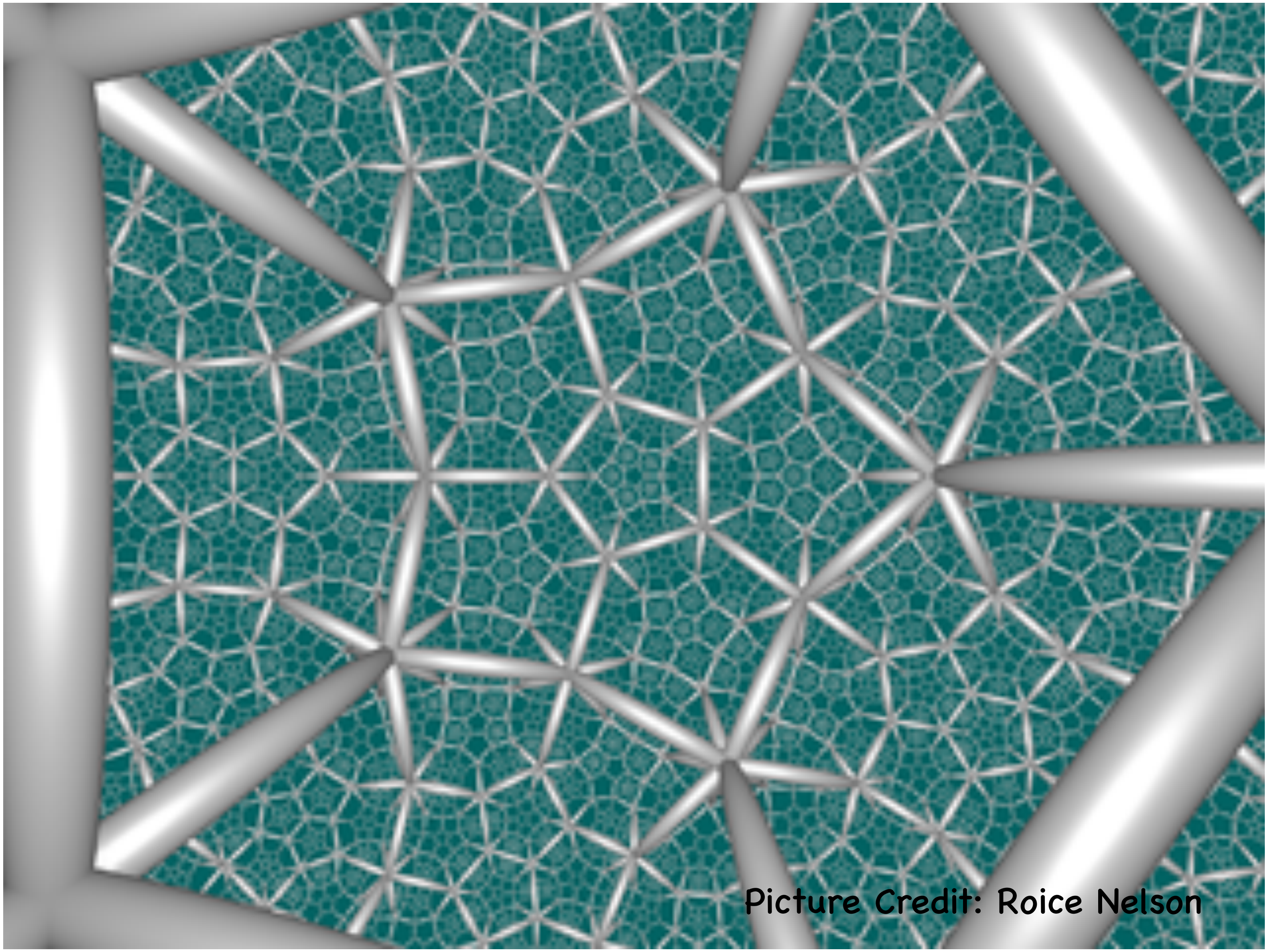
Picture Credit: Weeks 1985

The (5,3,3) tiling of the 3-sphere. There is one more infinite dodecahedron outside the shown figure.



Picture Credit: Weeks 1985

Seifert-Weber space: glue opposite faces using a 3/10 turn instead.



The (5,3,5) tiling of hyperbolic space covers the Seifert-Weber space.

There are 8 model geometries for 3-dimensional "manifolds":

Euclidean, Spherical, Hyperbolic 3-space

Sphere X line, Hyperbolic plane X line,

Nil Geometry, Sol Geometry,

Geometry of the universal cover of SL(2,R)

Recent result: Thurston's geometrization conjecture

Any compact 3-manifold without boundary can be decomposed into pieces so that each piece has one of the 8 model geometries.

This result implies the Poincaré conjecture, worth 1,000,000$ to solve.

شكرا جزيلا !

Some interesting links:

1. https://www.youtube.com/watch?v=c_O58ewaoPk

2. https://www.youtube.com/watch?v=AAsICMPwGPY

3. http://nilesjohnson.net/hopf.html

4. https://vimeo.com/47049144