SARSI 2016

First Week Lectures

Math - Kim Whittlesey

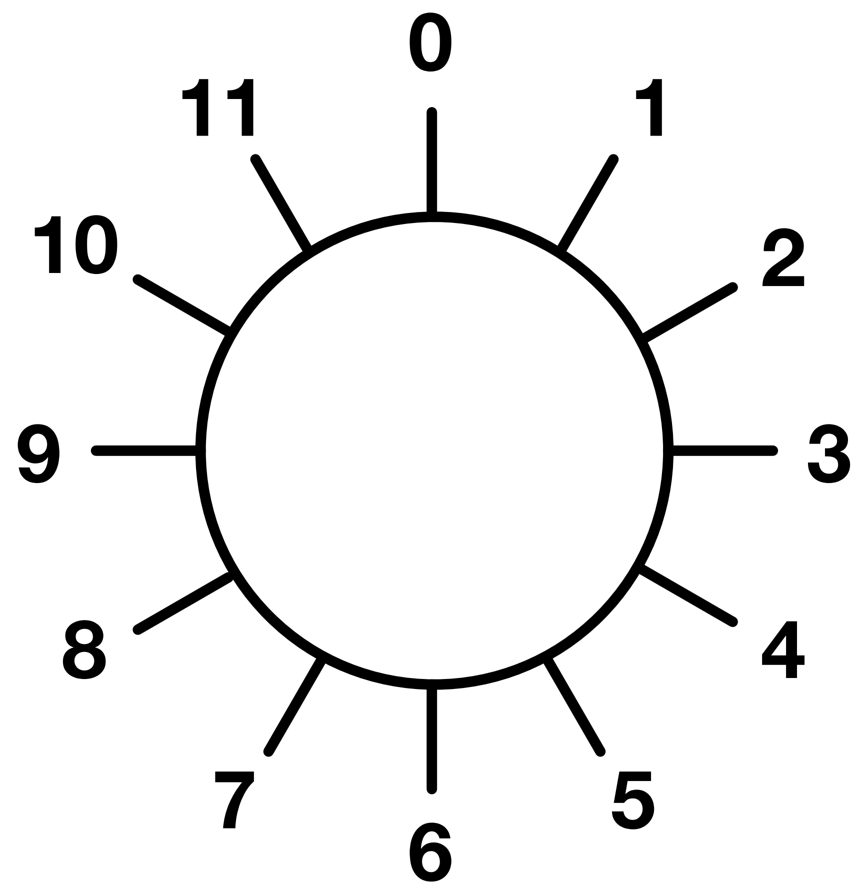
Lecture 4: Groups

مجموعة

Groups

Clock addition

Circle (clock) addition:



examples:

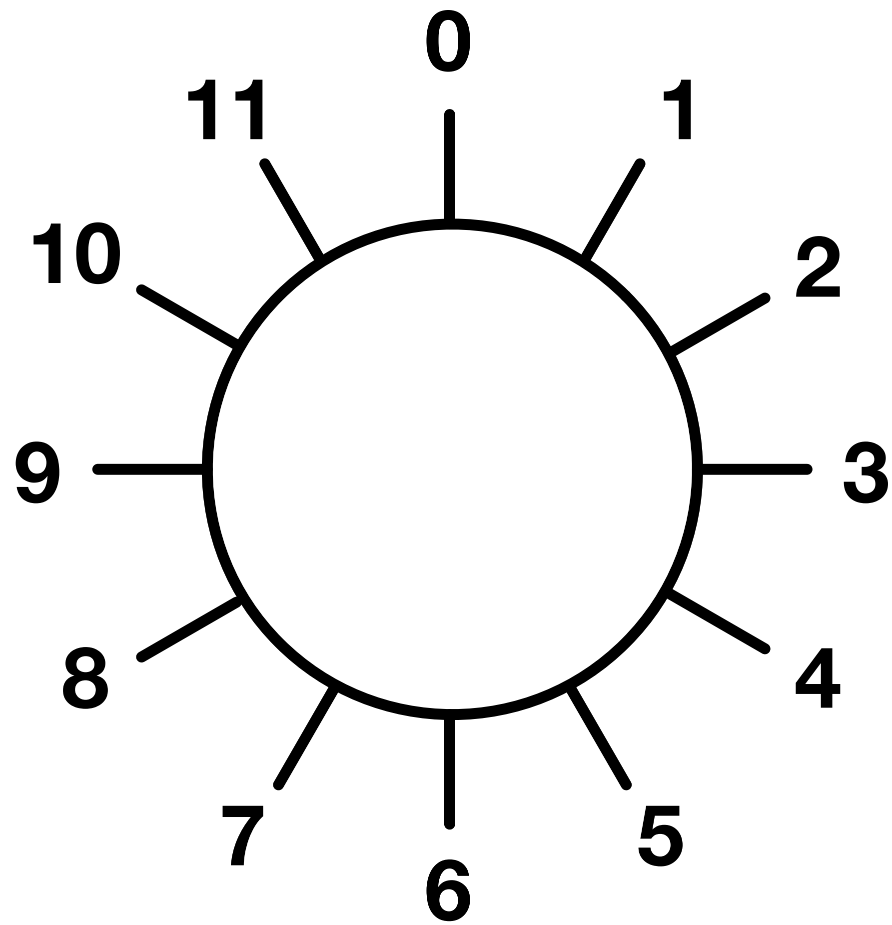
9 + 4 = 1

10 + 4 = 2

11 + 1 = 0

We have a set of numbers:
{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}.

We also have the operation "+".



+	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	3	4	5	6	7	8	9	10	11	0
2	2	3	4	5	6	7	8	9	10	11	0	1
3	3	4	5	6	7	8	9	10	11	0	1	2
4	4	5	6	7	8	9	10	11	0	1	2	3
5	5	6	7	8	9	10	11	0	1	2	3	4
6	6	7	8	9	10	11	0	1	2	3	4	5
7	7	8	9	10	11	0	1	2	3	4	5	6
8	8	9	10	11	0	1	2	3	4	5	6	7
9	9	10	11	0	1	2	3	4	5	6	7	8
10	10	11	0	1	2	3	4	5	6	7	8	9
11	11	0	1	2	3	4	5	6	7	8	9	10

This table shows all of the combinations.

In circle addition,

the number 0 is special:

$0 + 1 = 1$

$2 + 0 = 2$

and so on.

We say 0 is the **IDENTITY**

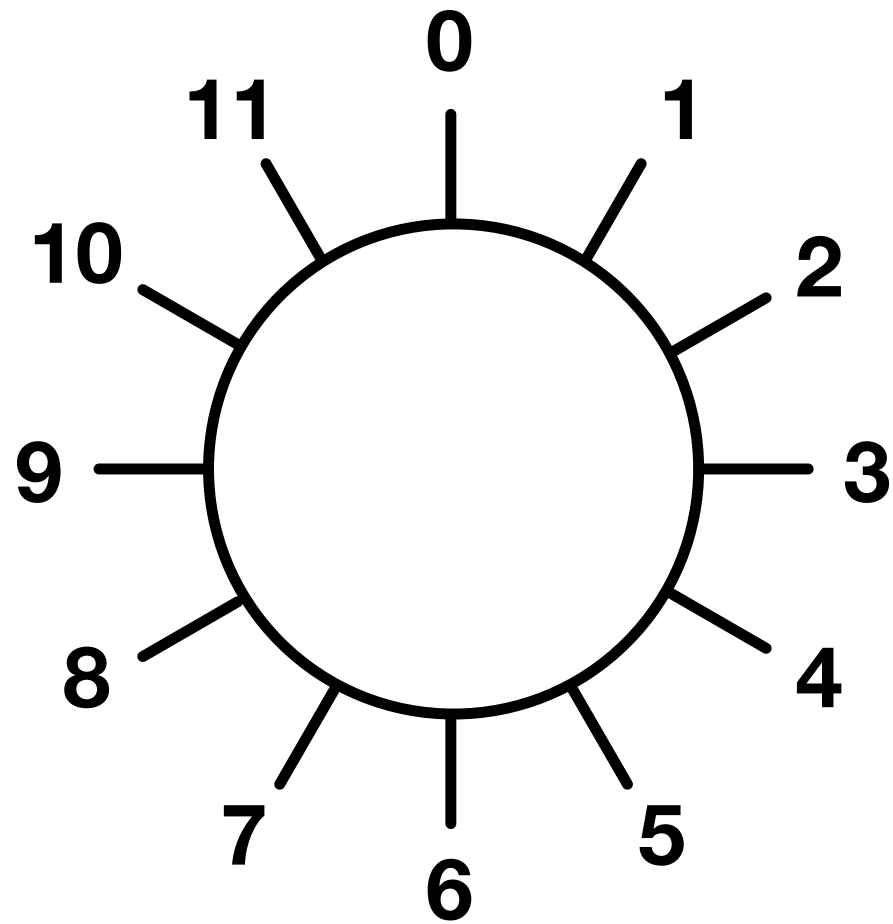
element.

Every number has an "opposite", or **INVERSE**:

10 + 2 = 0

5 + 7 = 0

3 + 9 = 0



Two elements are INVERSES if they combine to make the IDENTITY.

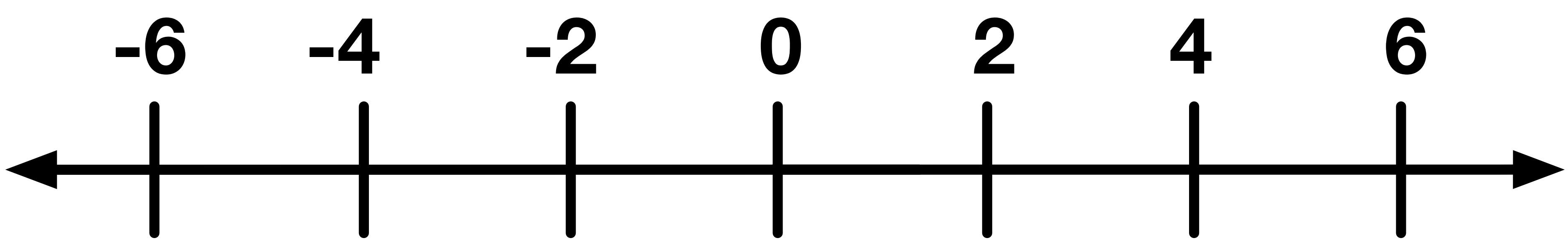
$$a + (-a) = 0$$

the Even Integers

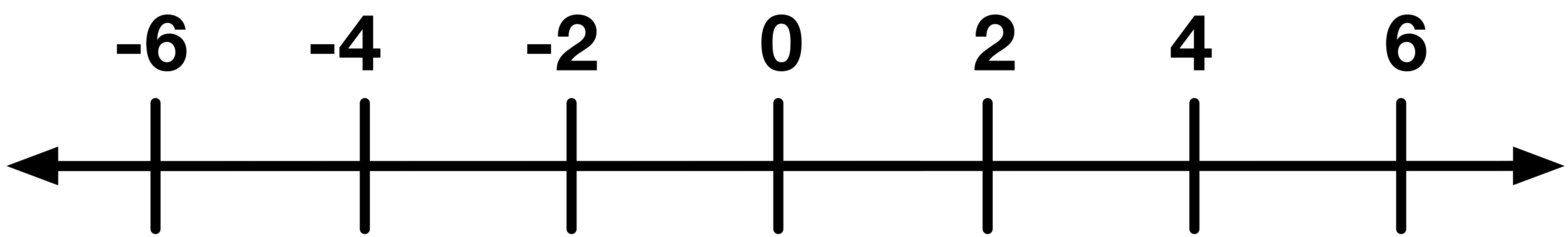
Set:

{..., −4, −2, 0, 2, 4, 6, 8, ... },

Operation: +



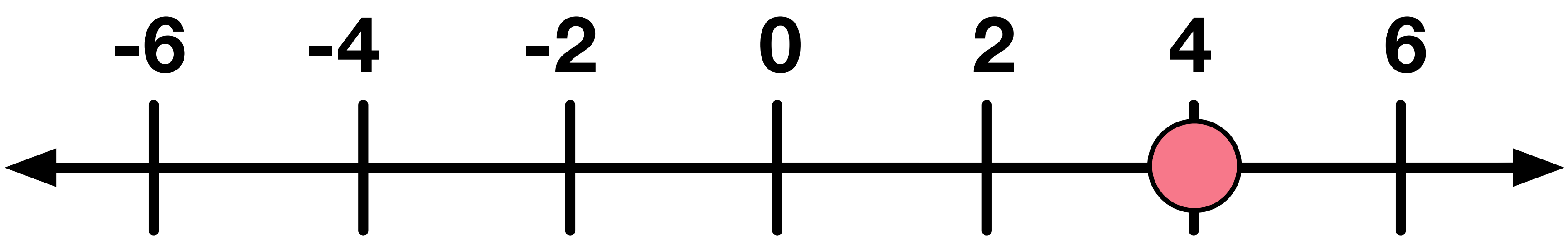
What is the IDENTITY element here?



What is the IDENTITY element here?

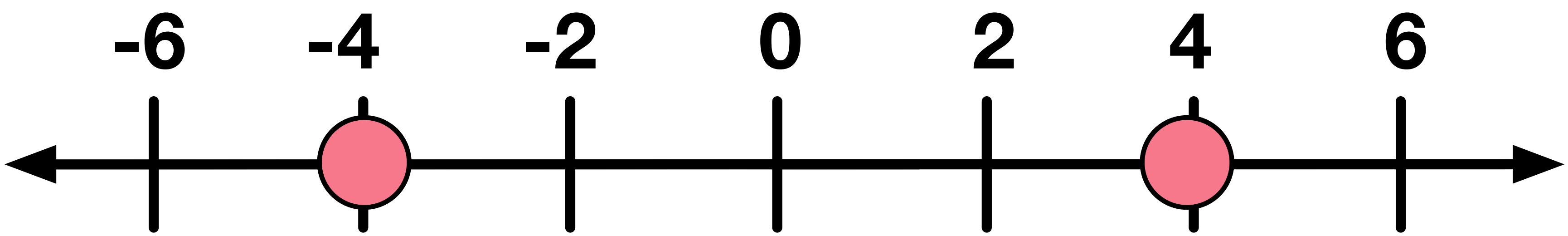
$$2+0 = 0+2 = 2$$

What is the inverse of 4?



What is the inverse of 4?

$$4 + (-4) = 0$$



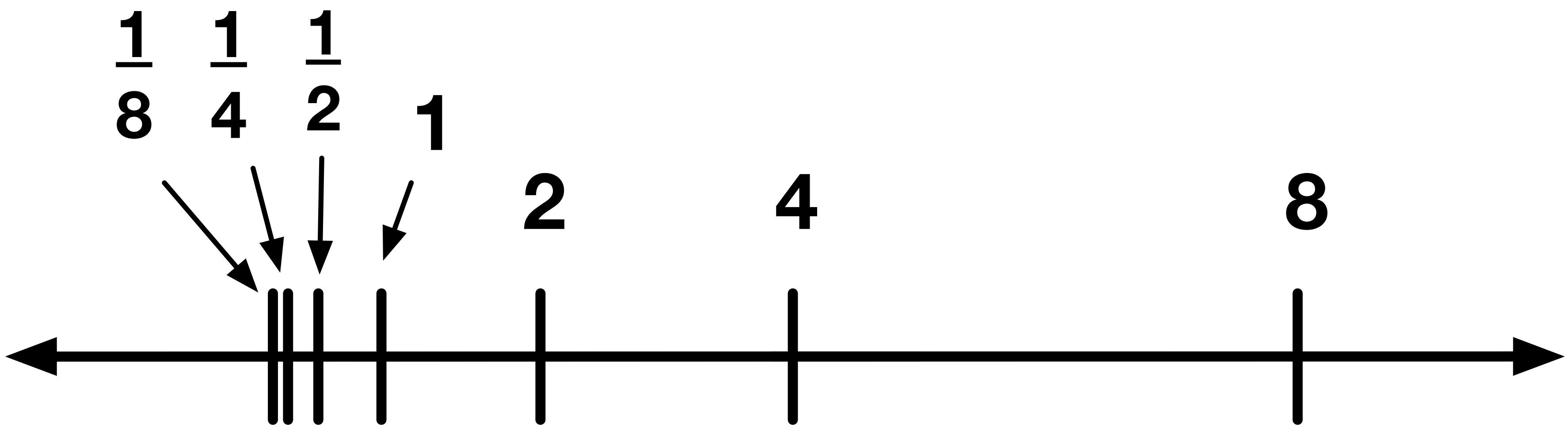
Is the sum of two even numbers always even?

Is the sum of two even numbers always even?

Yes: $2k + 2m = 2(k+m)$

New set: powers of 2

New operation: multiplication



Powers of 2:

{..., 1/8, 1/4, 1/2, 1, 2, 4, 8, ...}

Use the operation "x".

Problem: For the set of powers of 2, with multiplication:

A. Find the identity.

B. What is the inverse of 2^a?

C. Is the product of 2^a and 2^b also a power of 2?

What is the identity element?

What is the identity element?

$1 \times 2 = 2$

$1 \times 4 = 4$

$1 \times 1/2 = 1/2$

How can we find inverses?

How can we find inverses?

$2 \times 1/2 = 1$

$1/4 \times 4 = 1$

Is the product of two powers of two also a power of two?

Is the product of two powers of two also a power of two?

2 x 8 = 16

1/2 x 4 = 2

Is the product of two powers of two also a power of two?

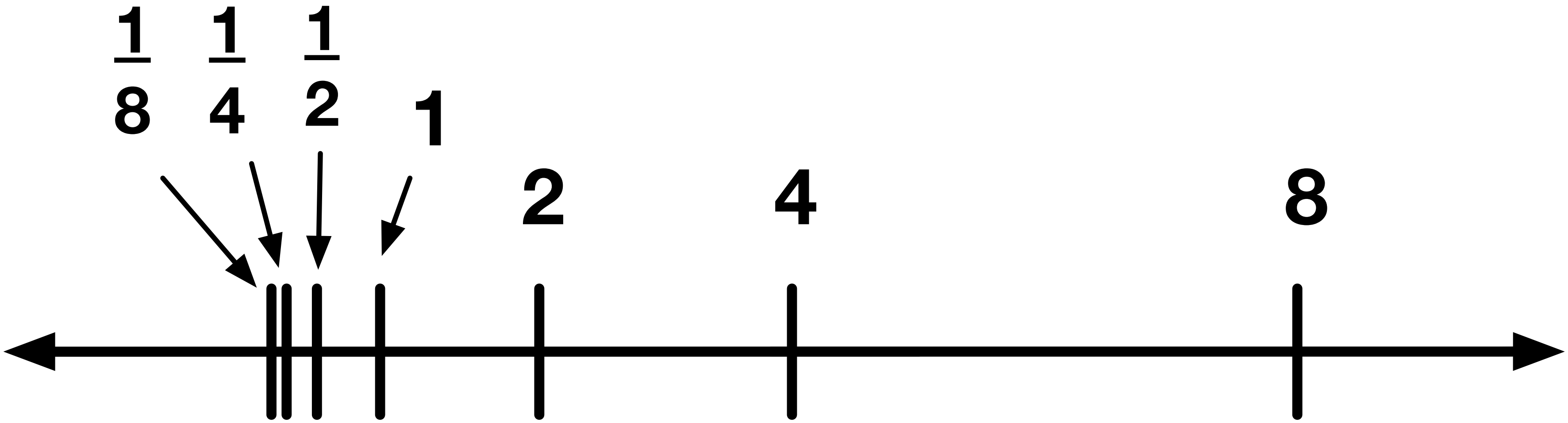
$$2 \times 8 = 16$$

$$1/2 \times 4 = 2$$

$$2^a \times 2^b = 2^{a+b}$$

There is also one more technical condition we need: the operation must be associative:

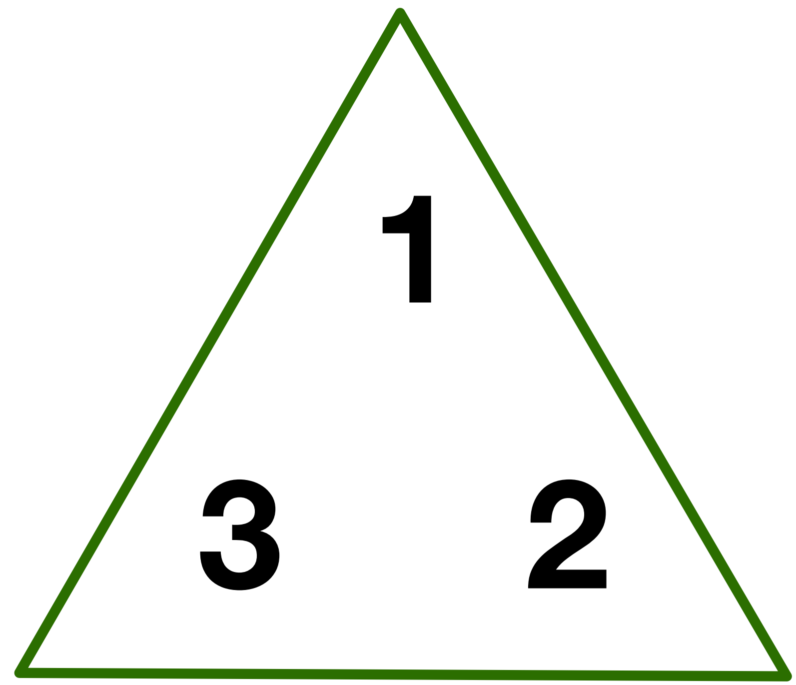
$$A * (B * C) = (A * B) * C$$



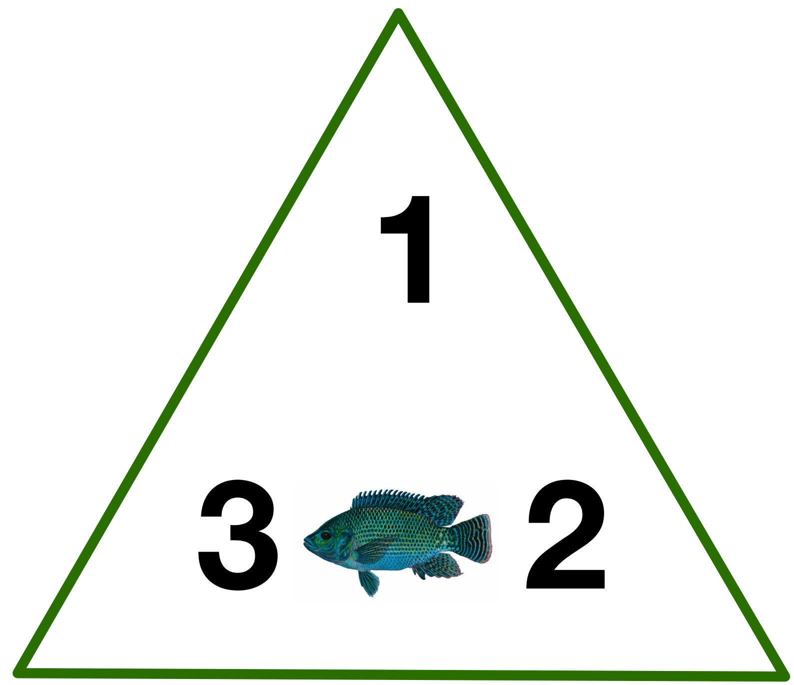
The set of powers of 2, with the operation "x", forms a GROUP.

We can also “multiply” things other than numbers.

Let's look at symmetries of an equilateral triangle.

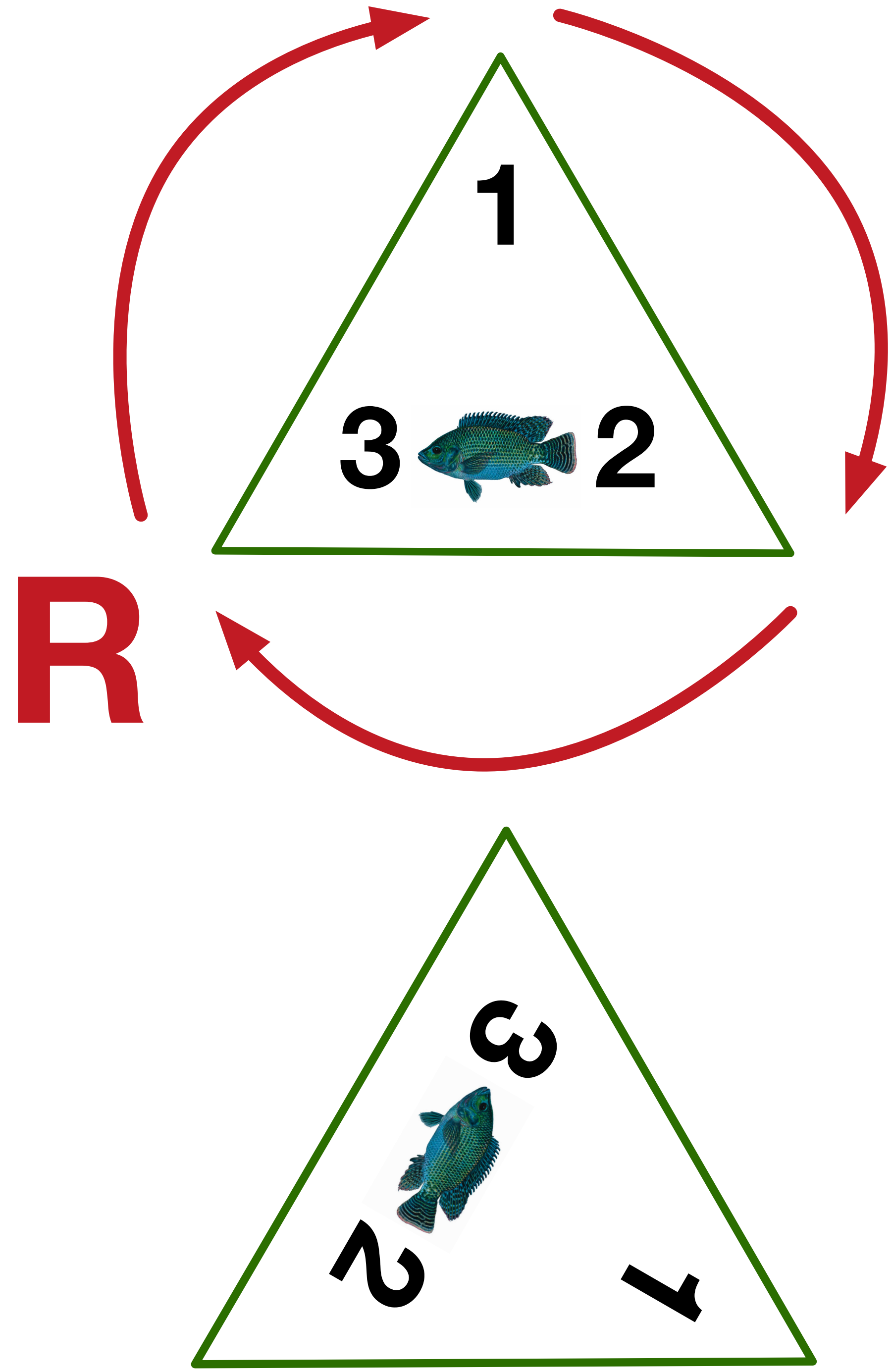


Cut out an equilateral triangle, label the corners on the front and back, and draw something.

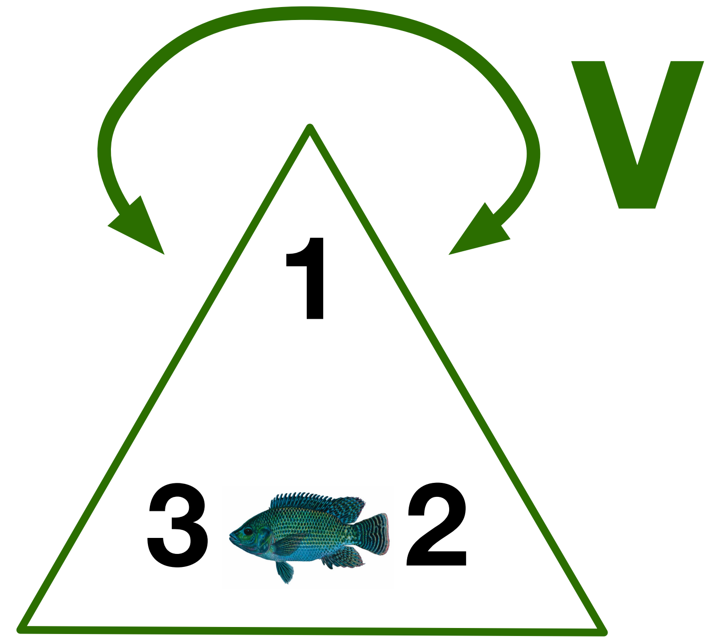


You can rotate the triangle by 120° clockwise.

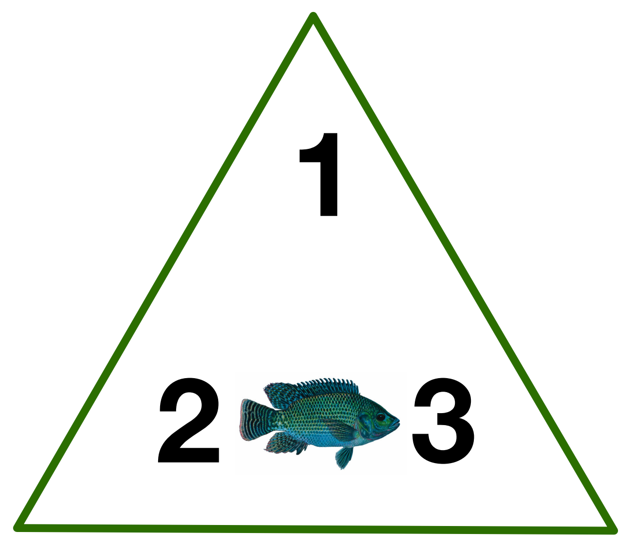
Call this symmetry R.



You can also flip the triangle across a vertical line.

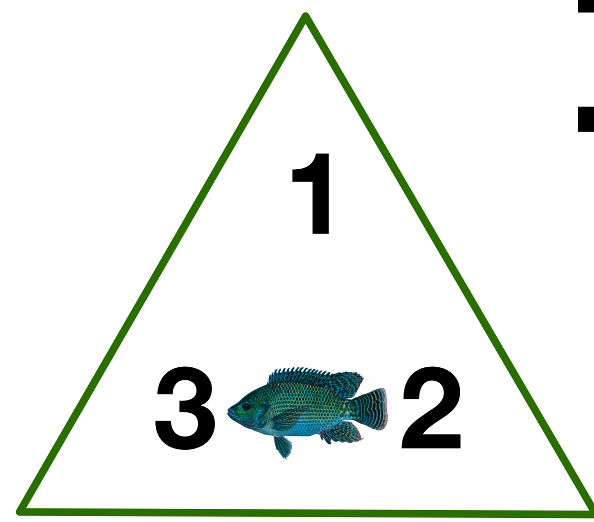


Call this symmetry V.

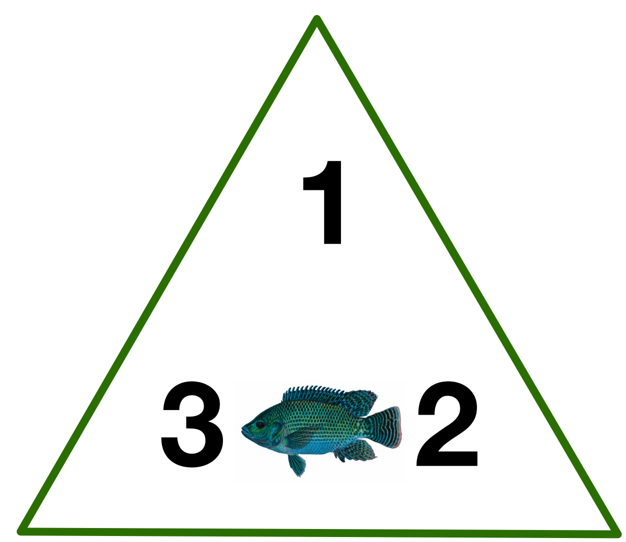


You can also just leave the triangle alone.

I

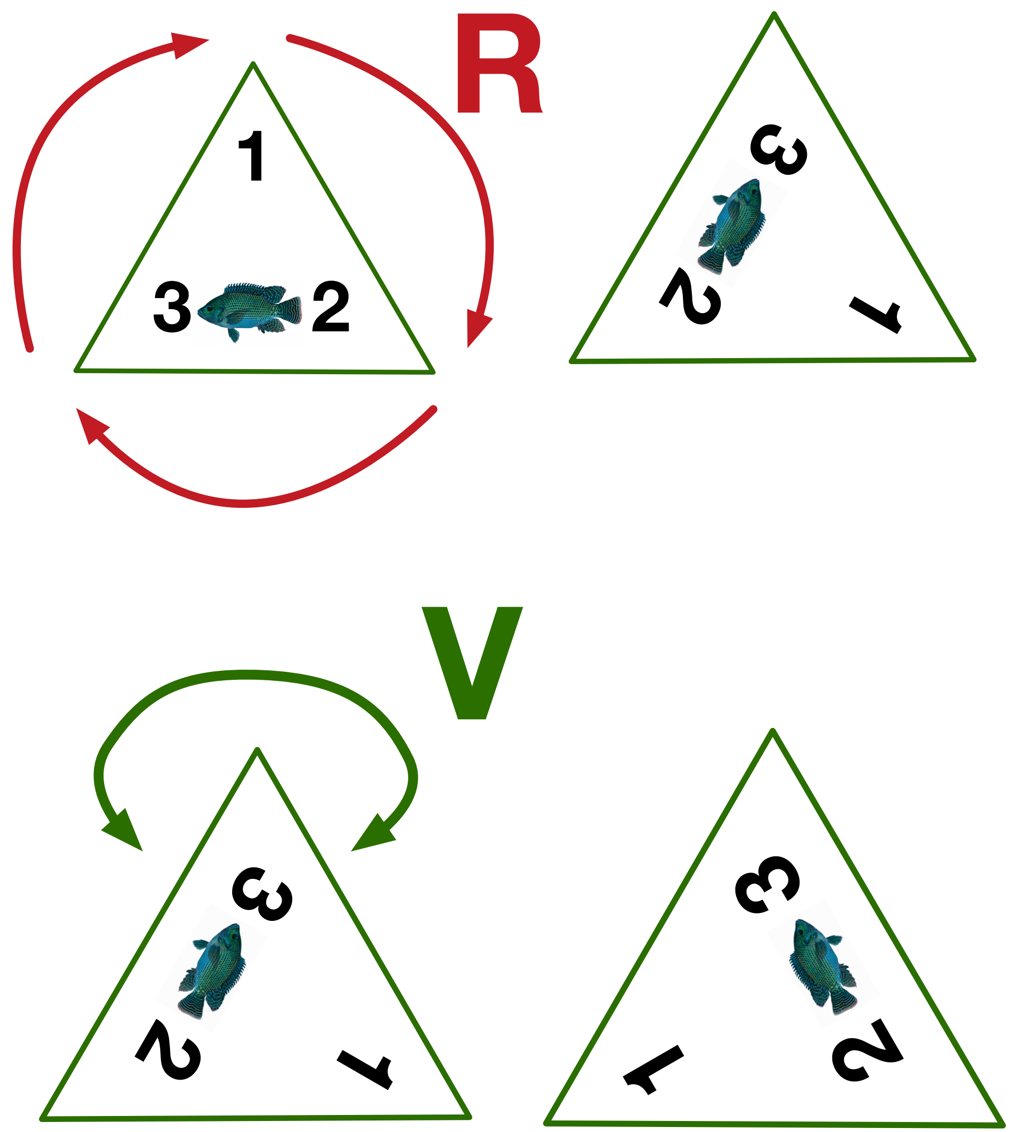


Call this symmetry I. (This is the identity.)



What happens if we first do R and then V?

We call this new symmetry R*V



True or False?

R*R*R = I

V*V = I

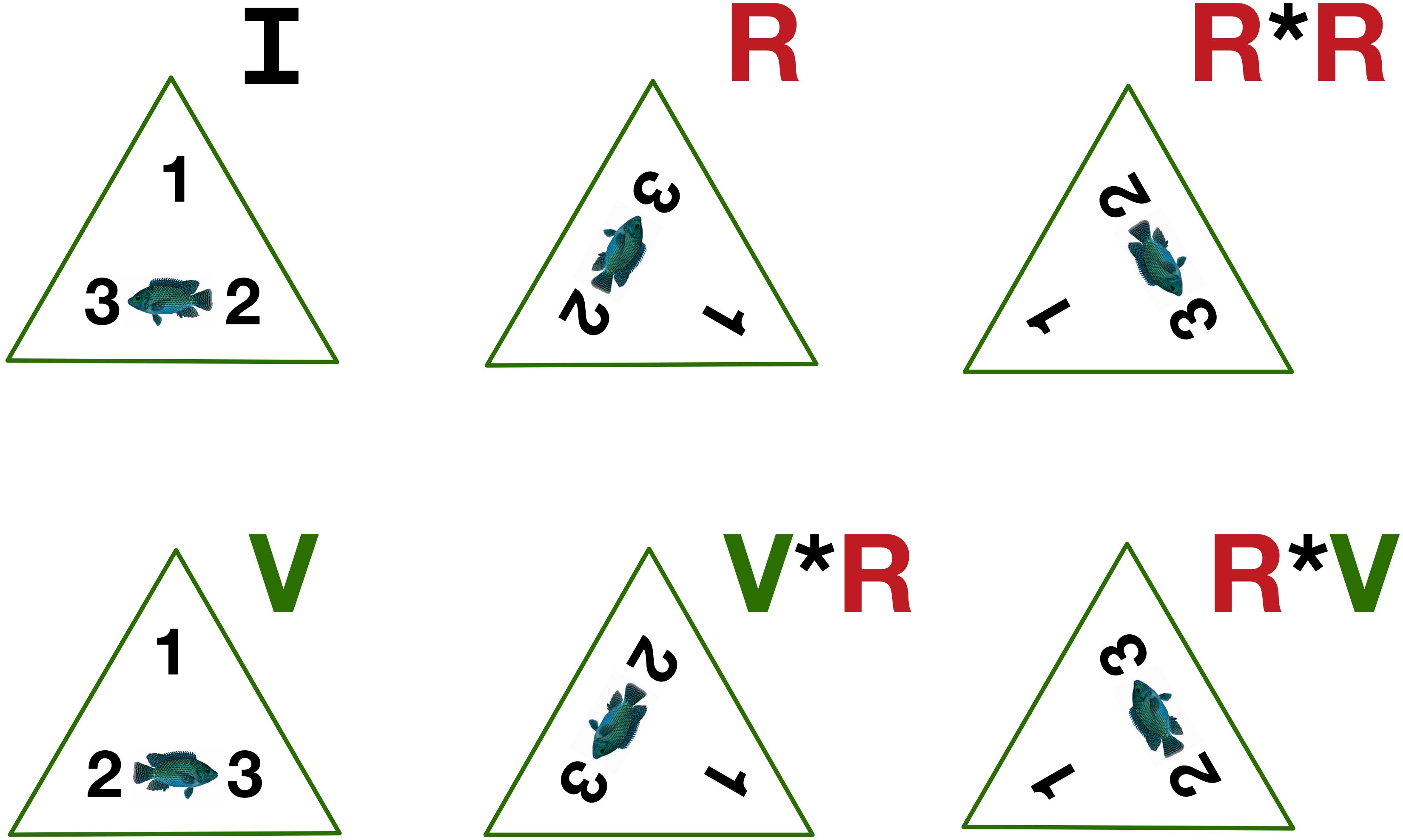
R*V = V*R

True or False?

$$R*V = V*R*R$$

Problem: Draw all of the symmetries of the equilateral triangle.

Label the pictures as products of V and R.



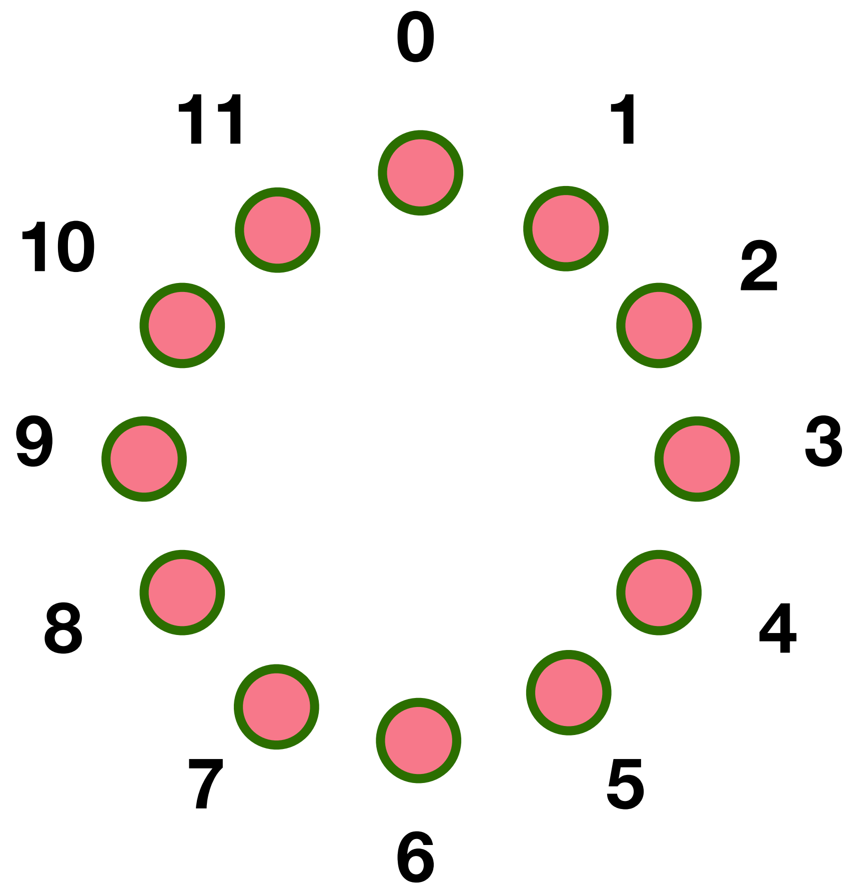
We have R*R*R = I, V*V = I,

and R*R*V = V*R

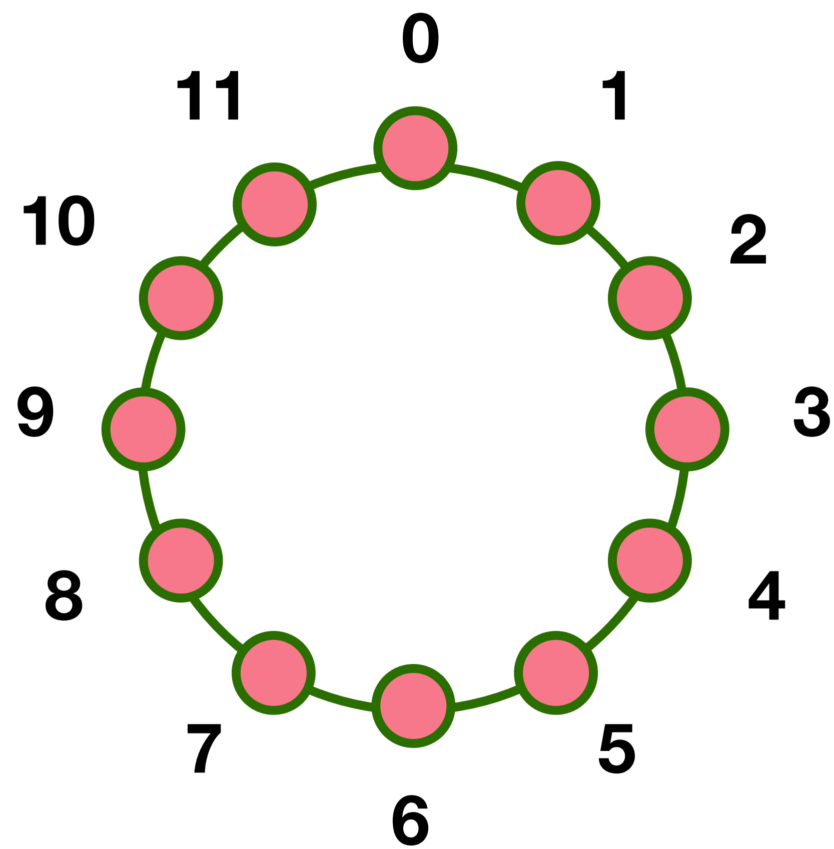
How to draw pictures of groups.

Circle addition:

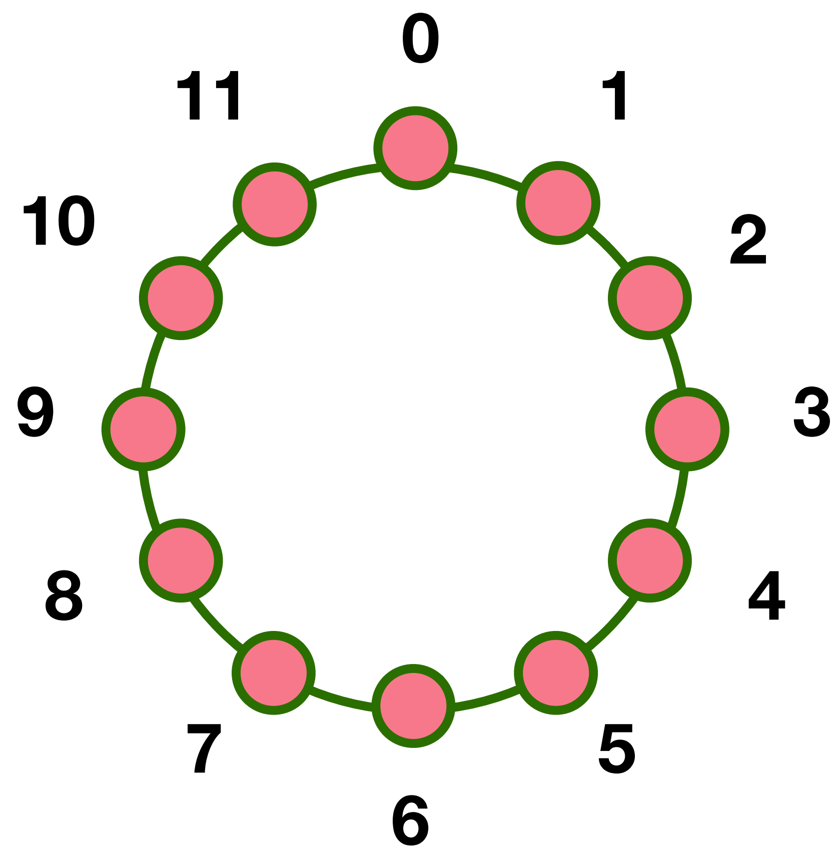
Draw a dot for each element of the group.

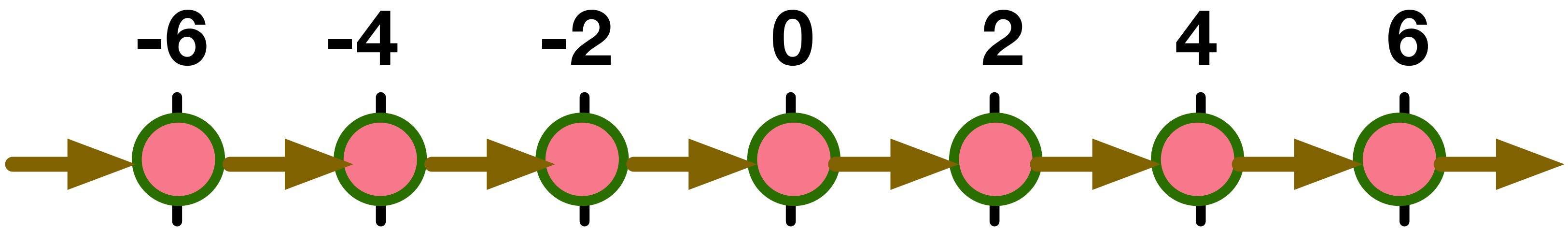


Connect elements p and q if $1 + p = q$.



We say 1 is a **GENERATOR** since we can get to ANY element by adding or subtracting enough 1's to the identity.

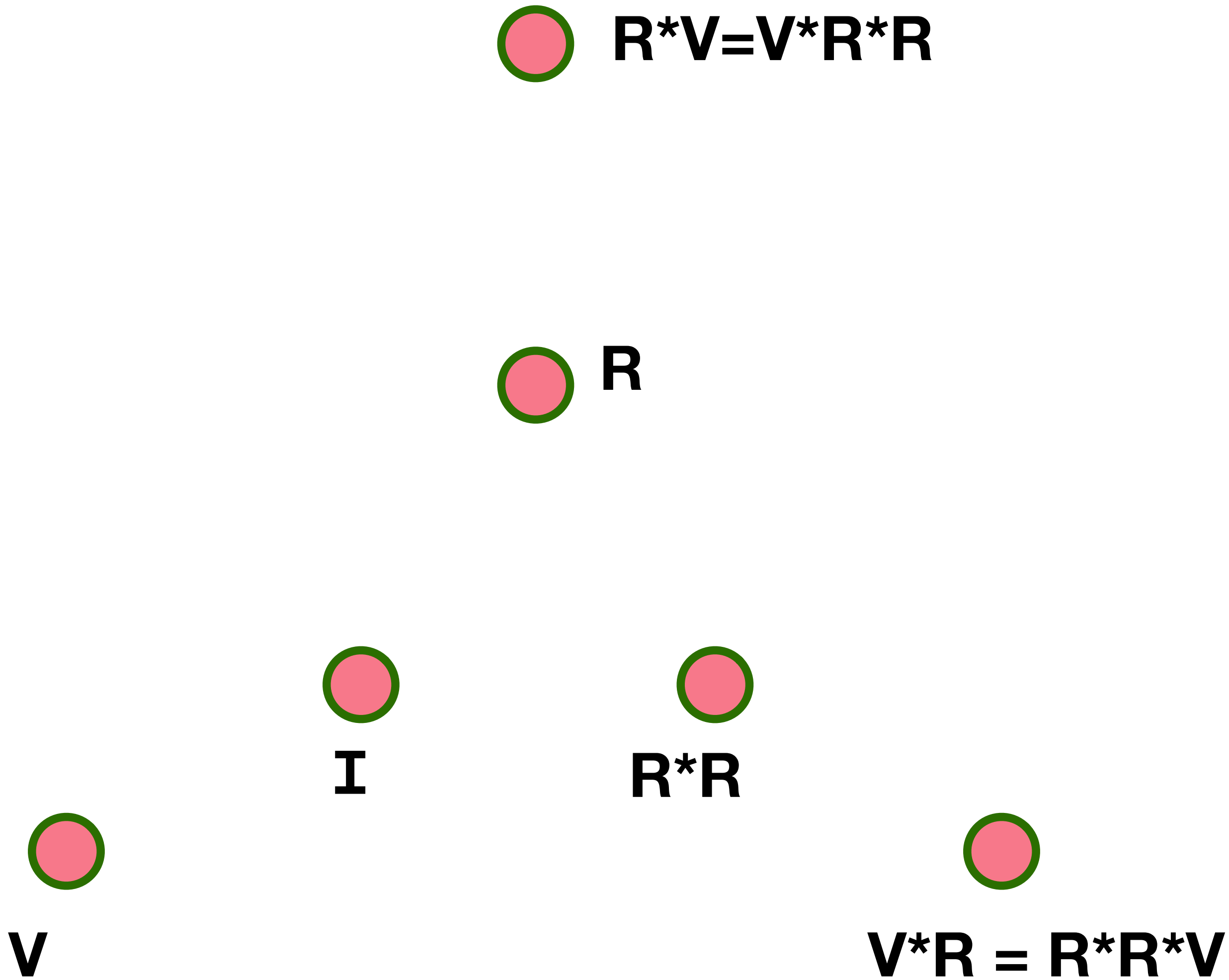




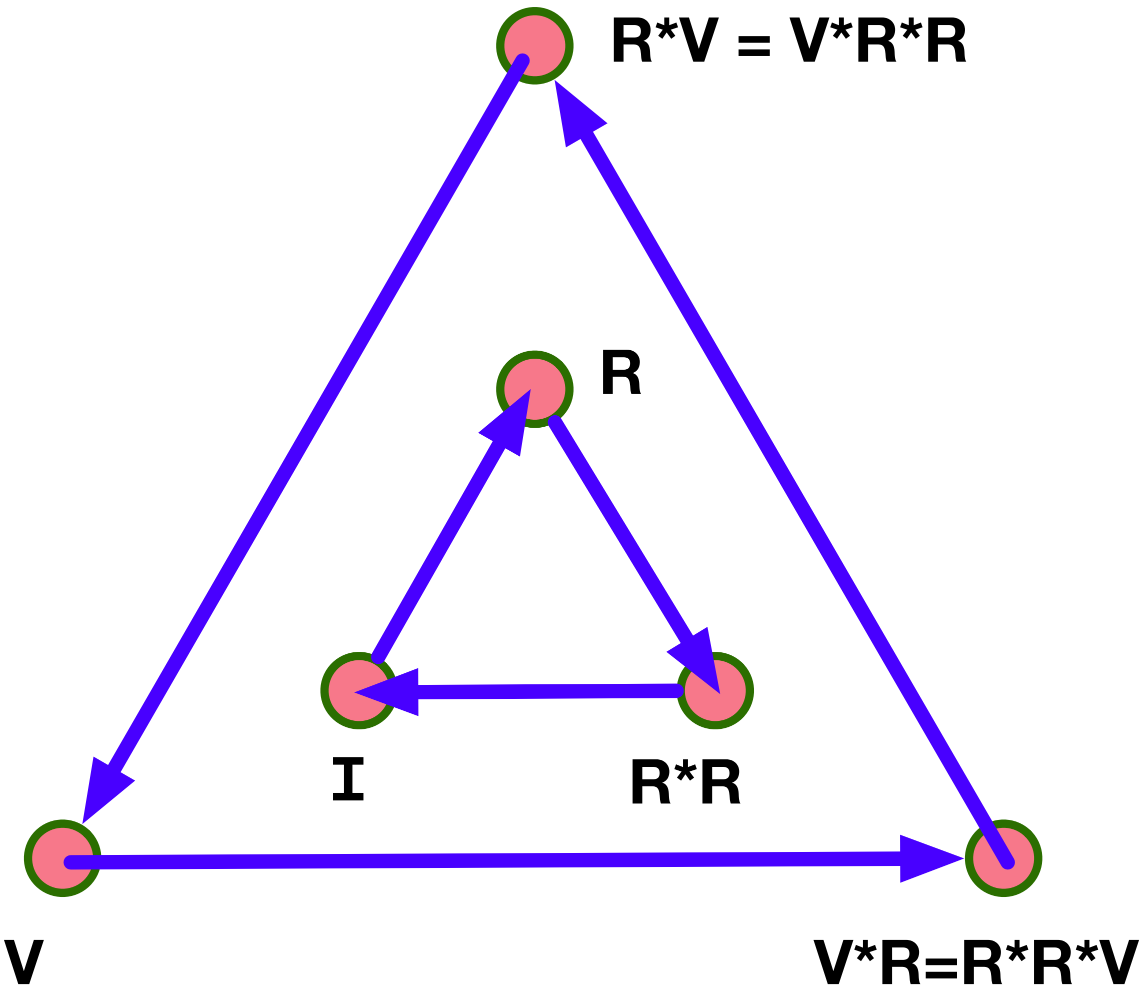
Graph of the even integers.

The generator is 2.

Lets draw the graph for the symmetries of the triangle.

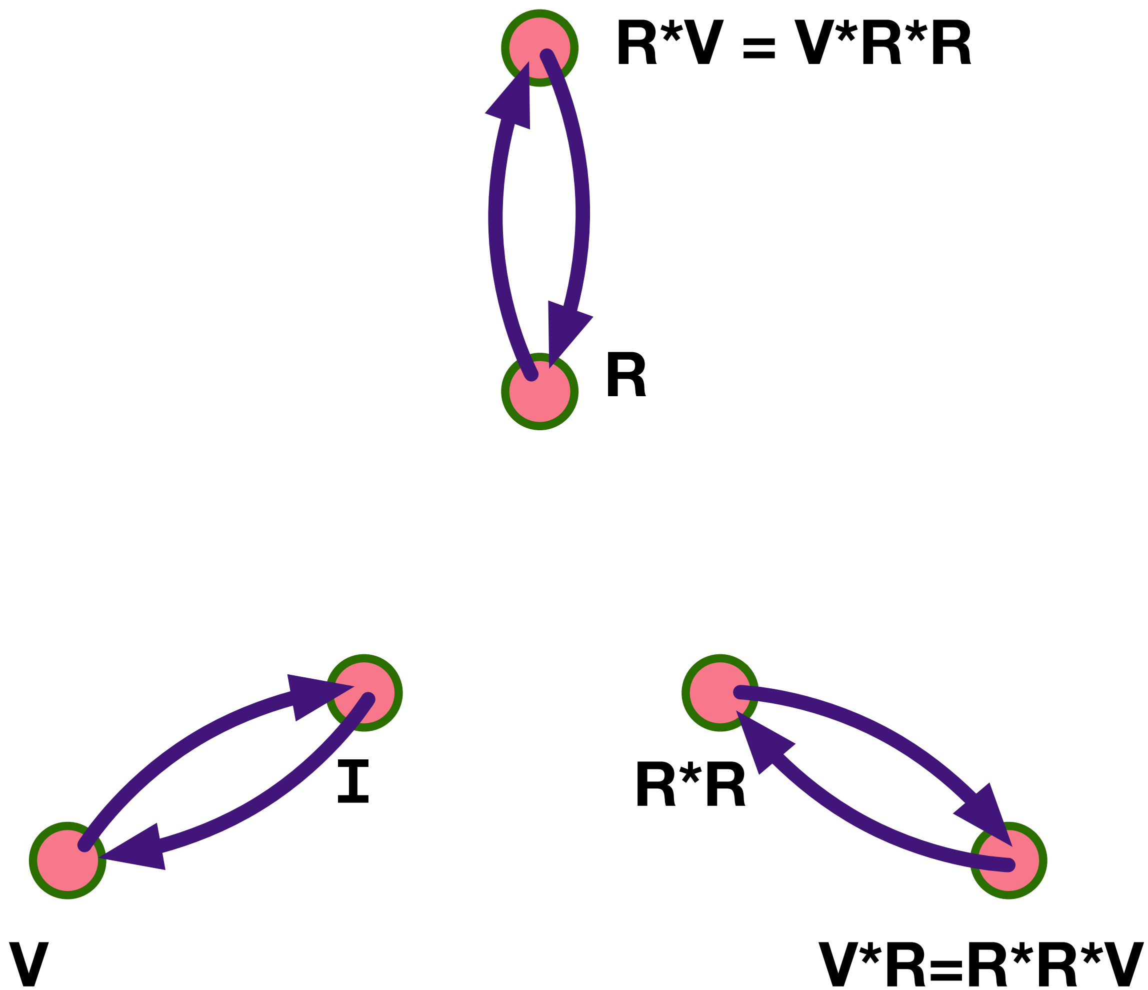


Symmetries of the triangle.



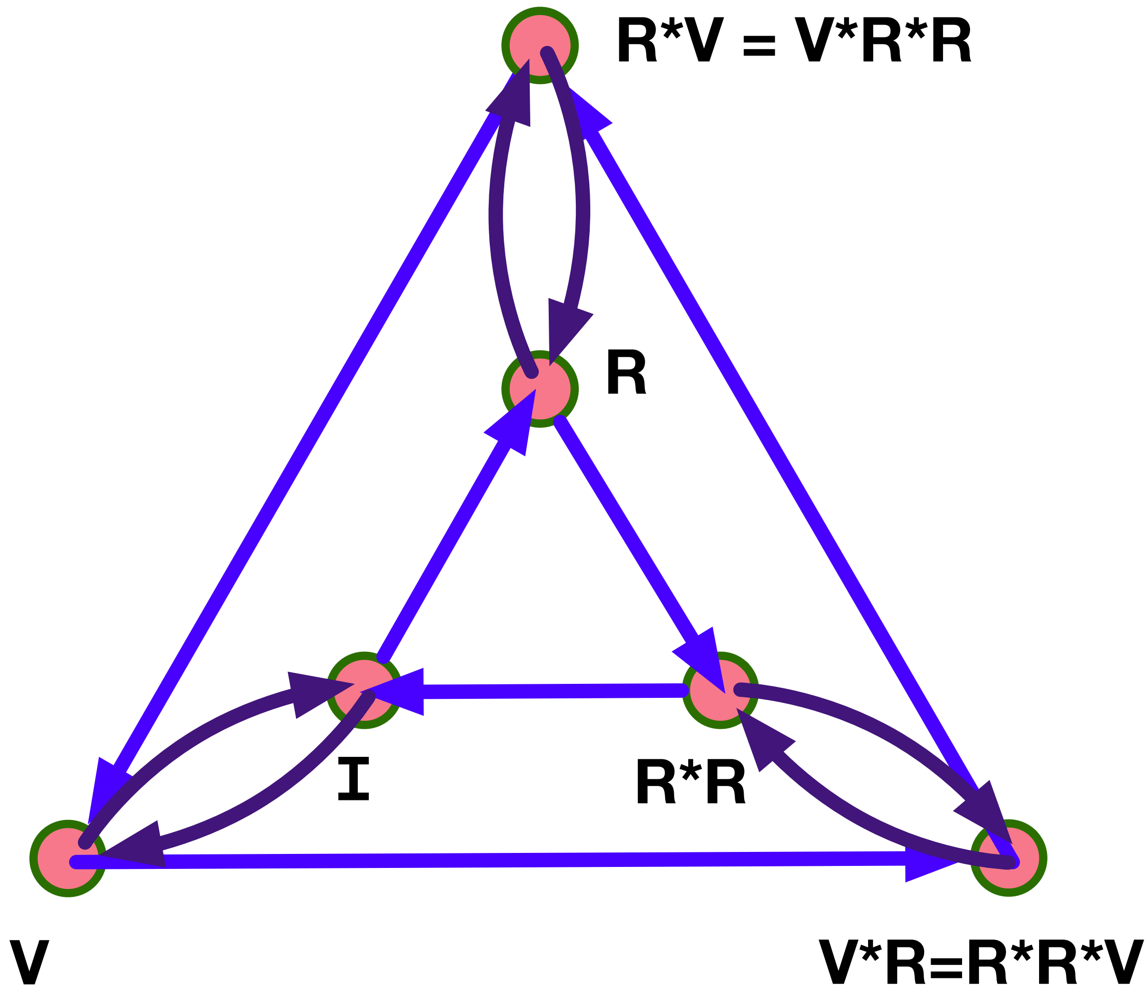
The edges for generator R.

Example: edge from V to R*V

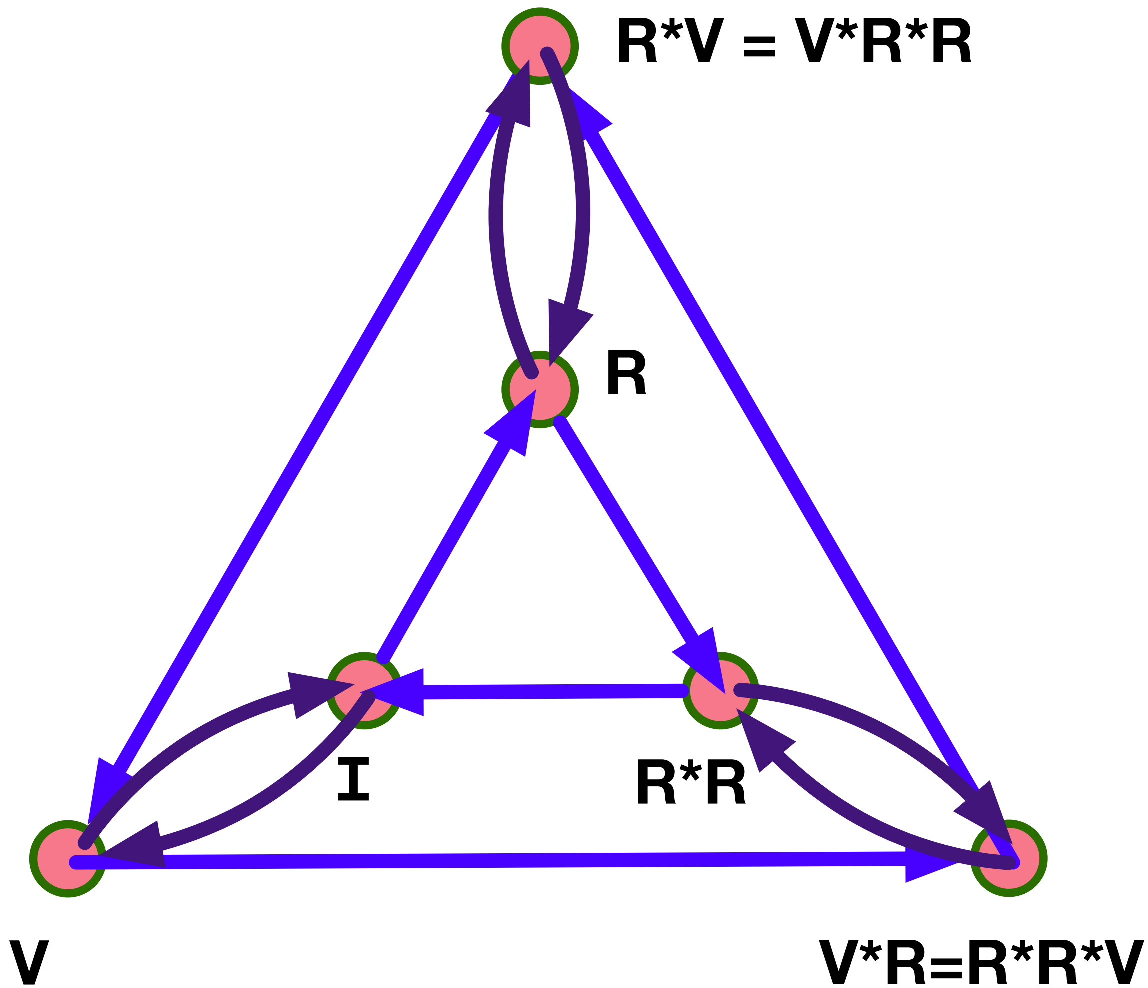


The edges for generator V.

Example: edge from R to V*R

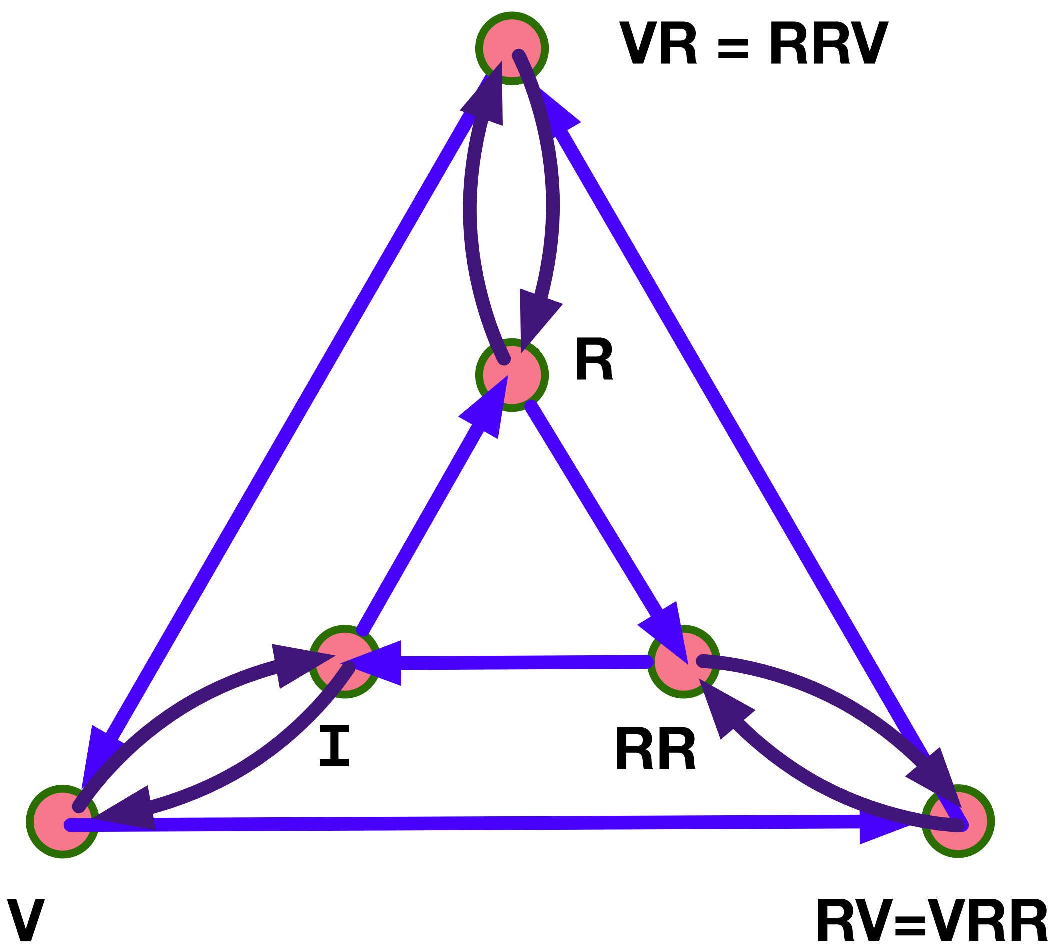


The entire graph.



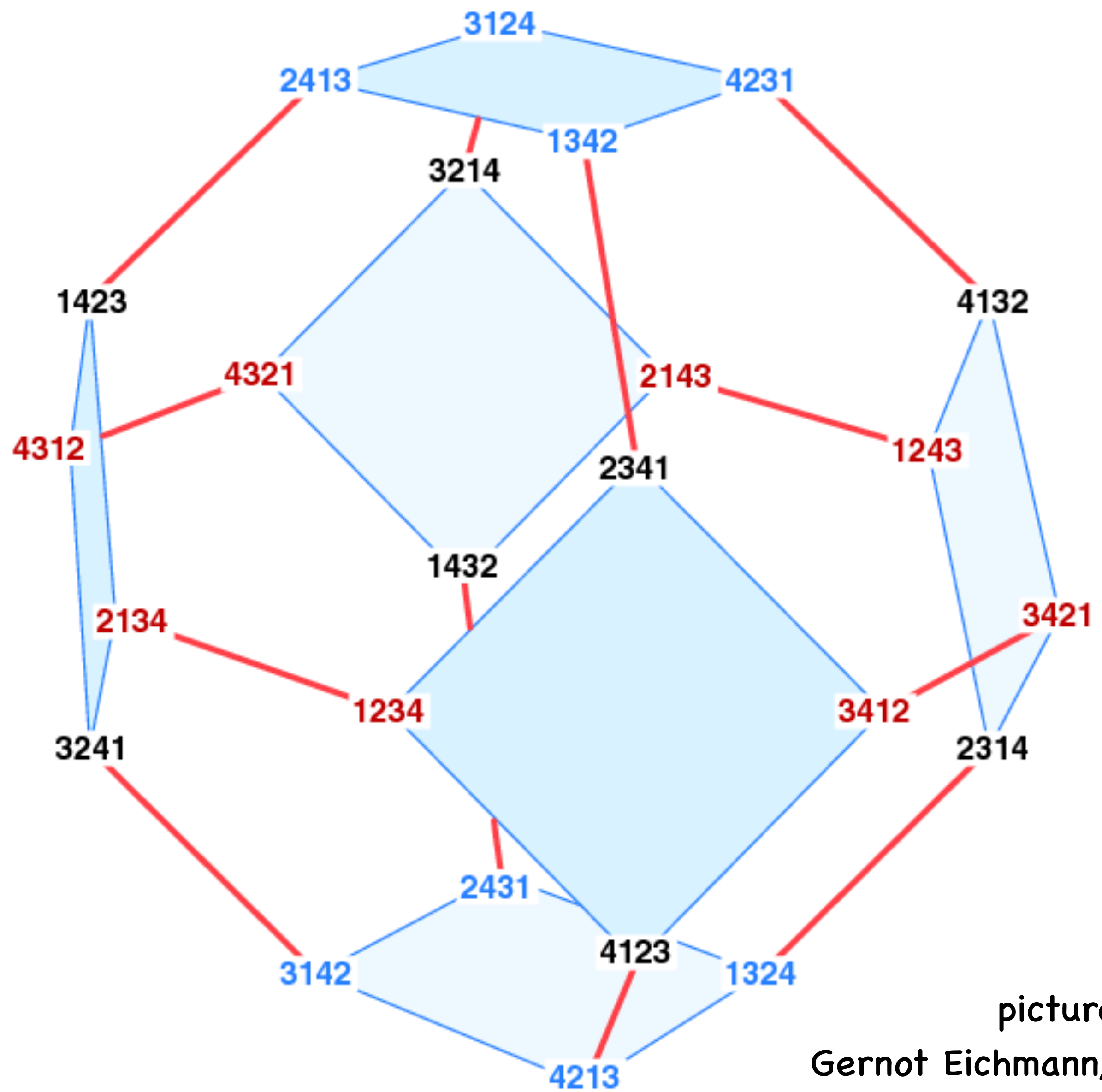
Problem: What is VVRRVRVVR equal to?

One method: use the equations to simplify VVRRVRVVR.



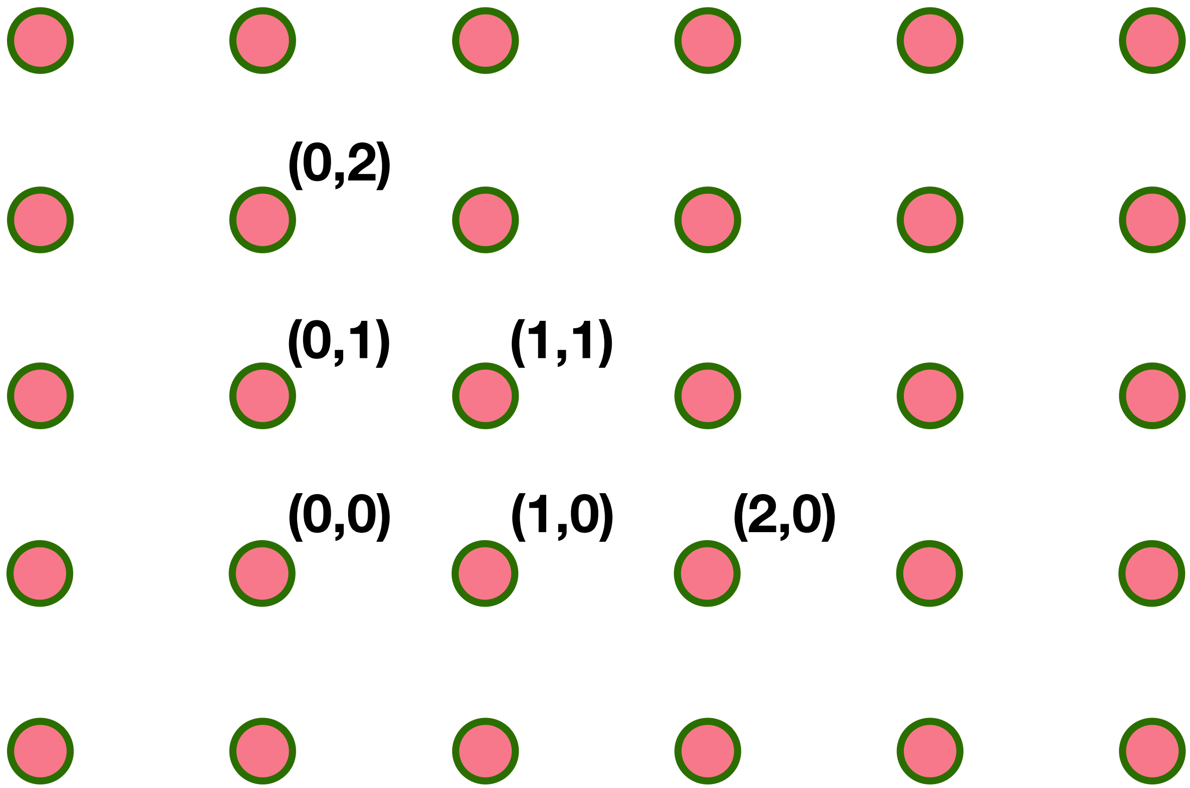
Or: trace through the graph to see VVRRVRVVR = V.

Some More Groups



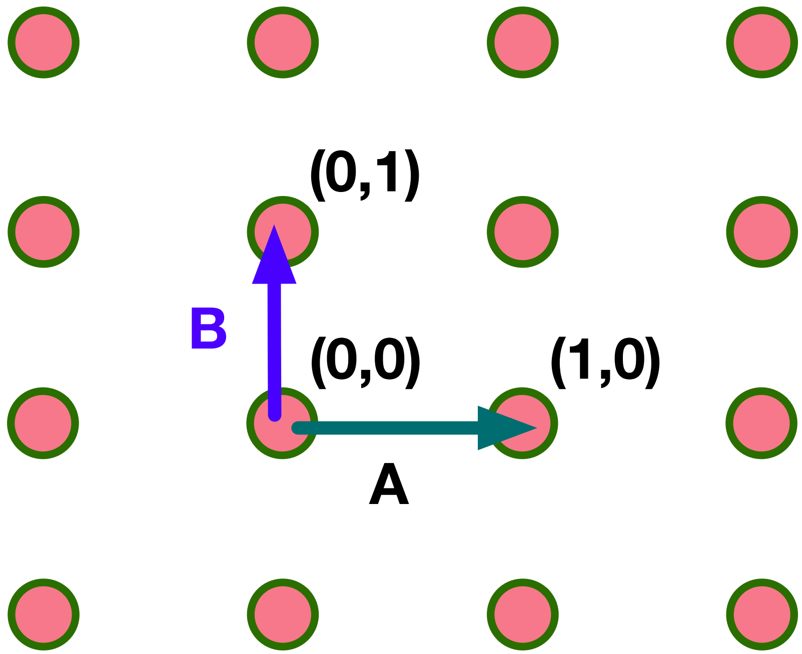
picture:
Gernot Eichmann, et al, 2015

Permutations of (1,2,3,4)



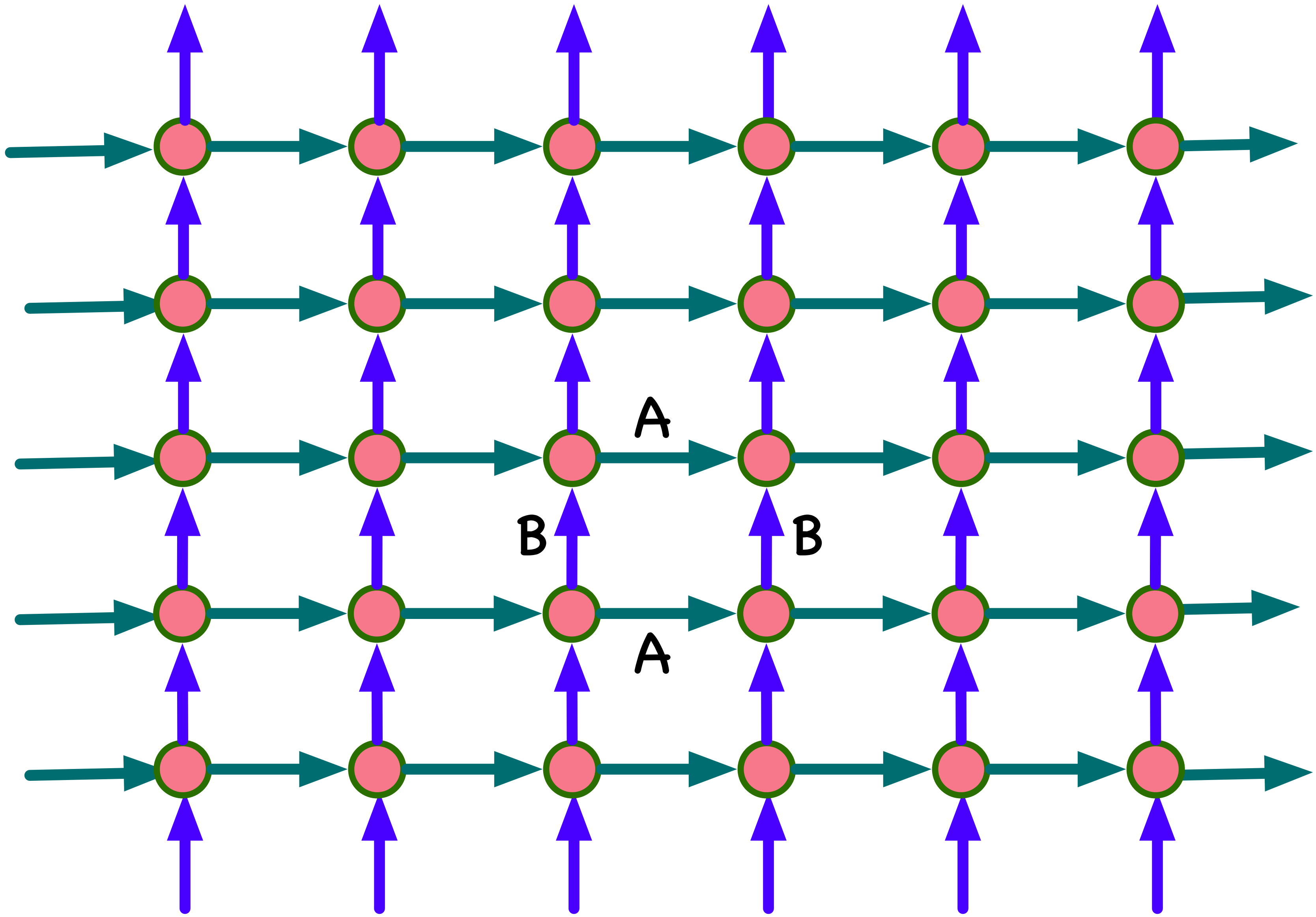
“integers x integers”

Pairs of integers (x,y), with addition.

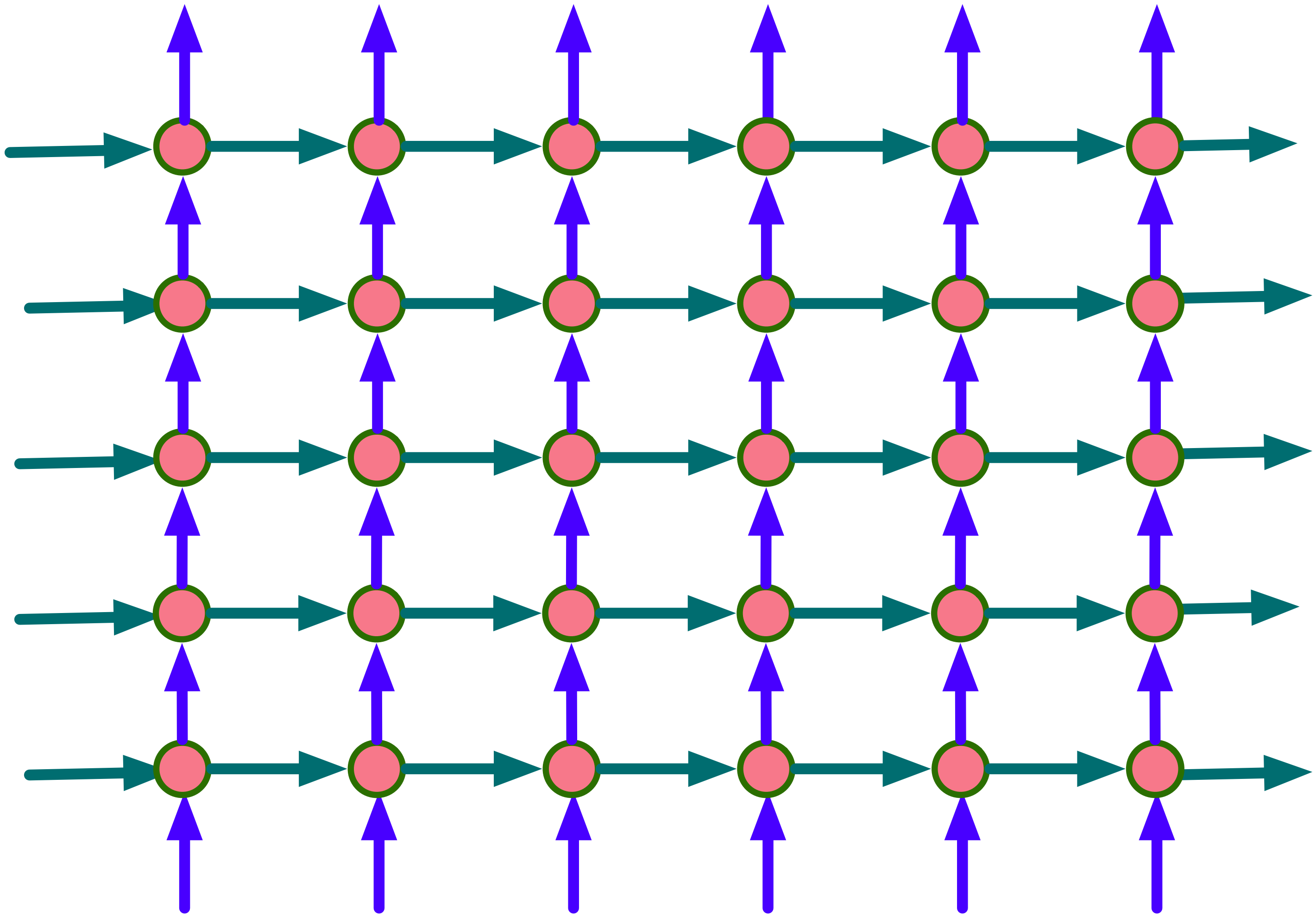


Use A = (0,1) and

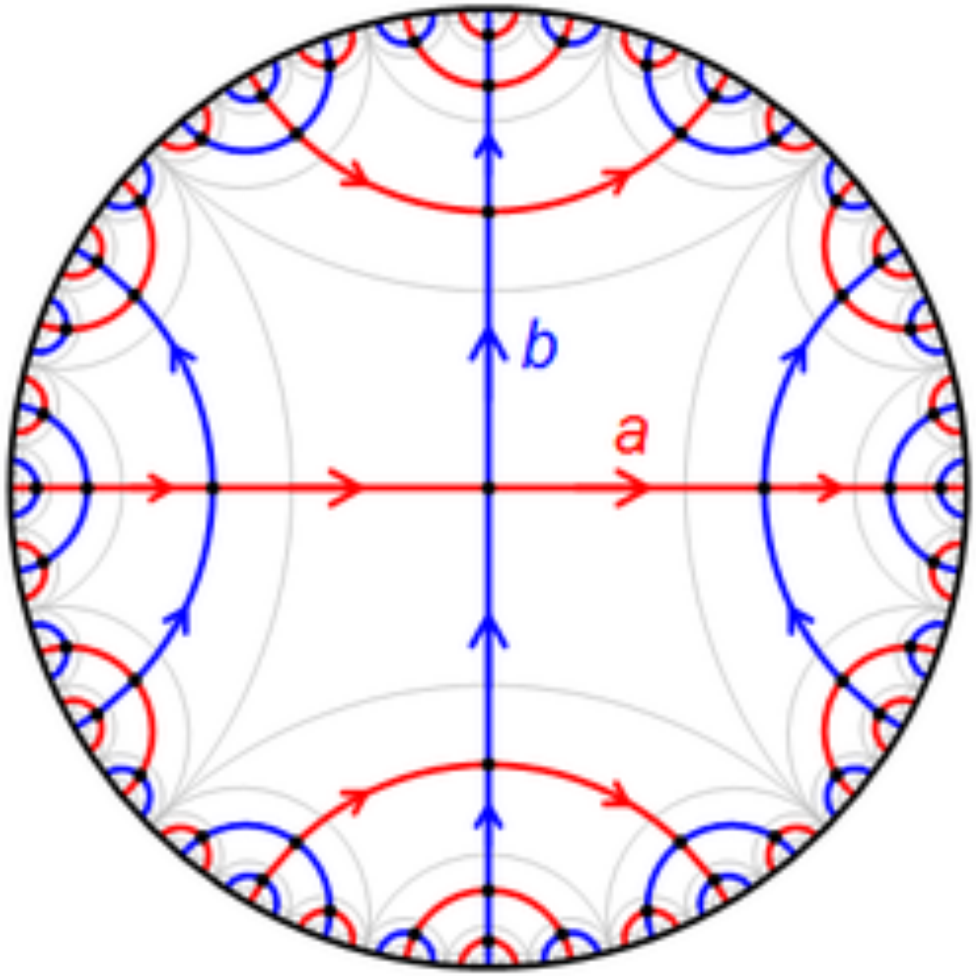
B = (0,1) as generators.



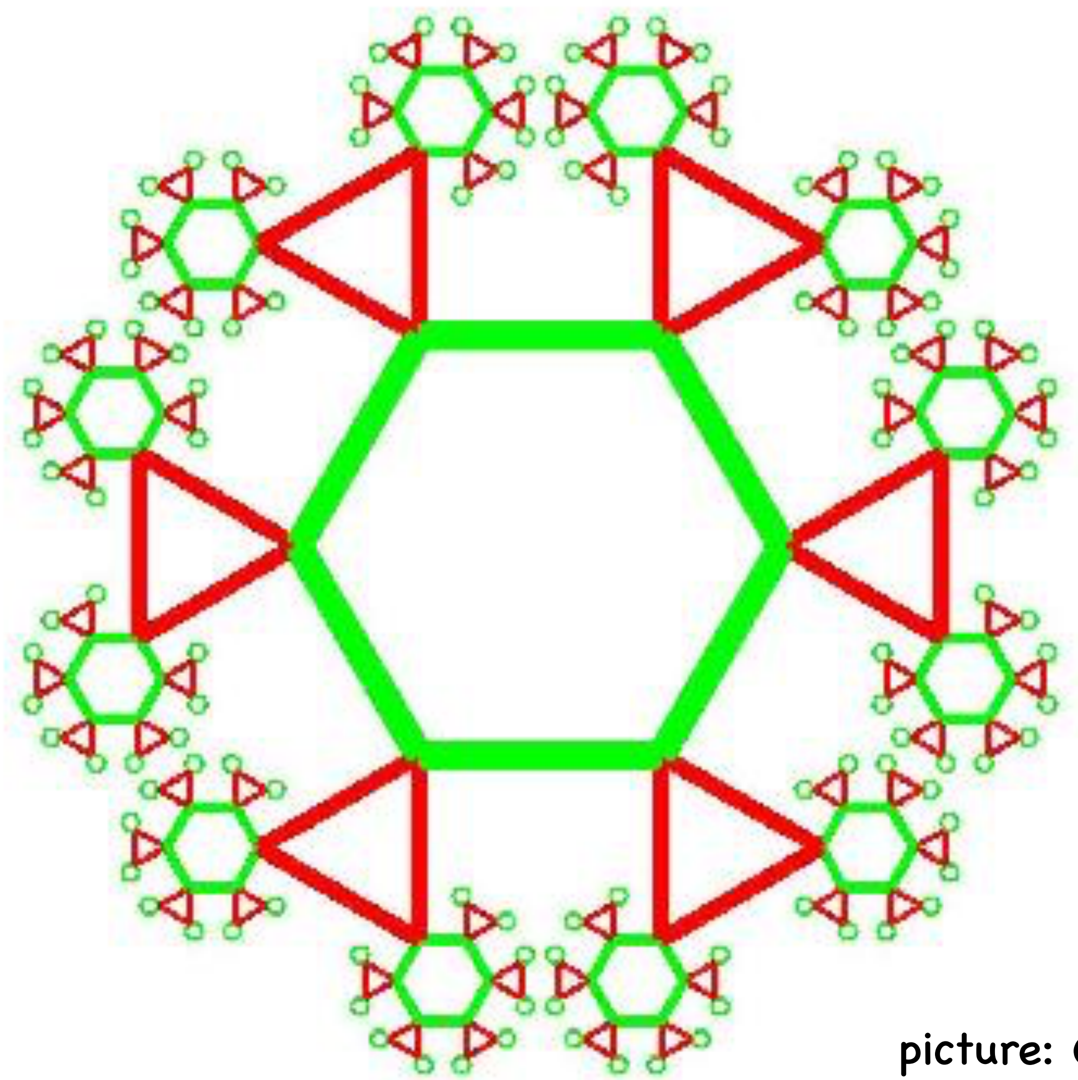
Notice that AB = BA



What kind of geometry does this look like?



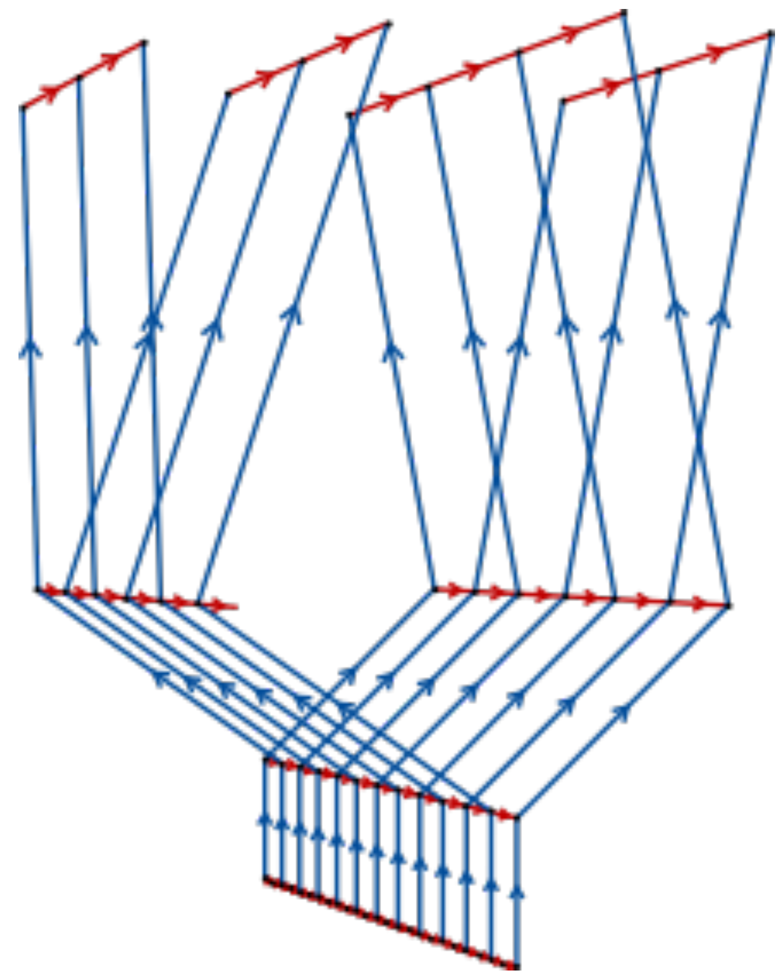
This group has generators A and B, but no equations.



picture: Cliff Reiter

This group has generators C and D, where CCCCCC = I and DDD = I

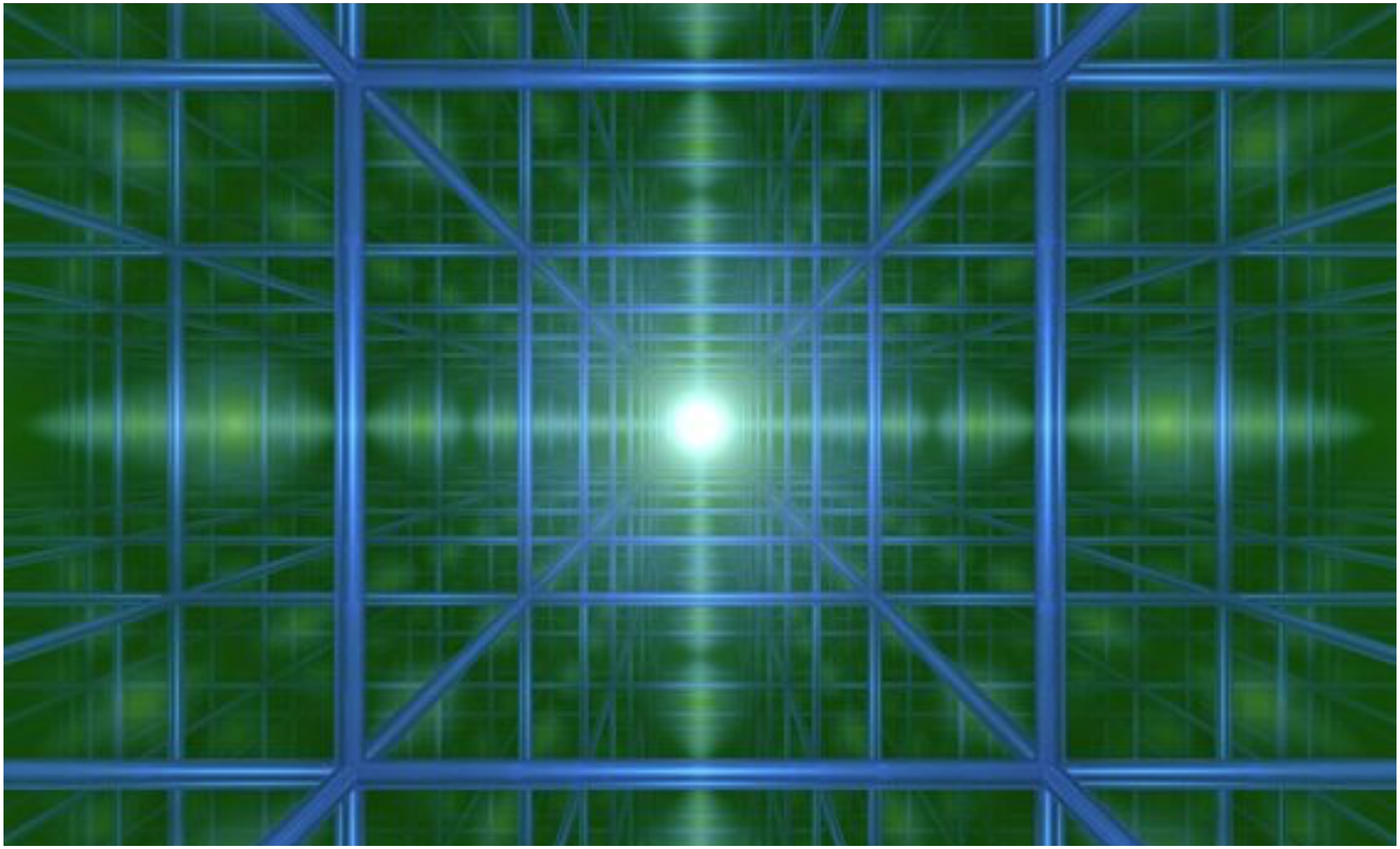
Most groups do not actually fit in the plane.



Generators: T,A

Equation:

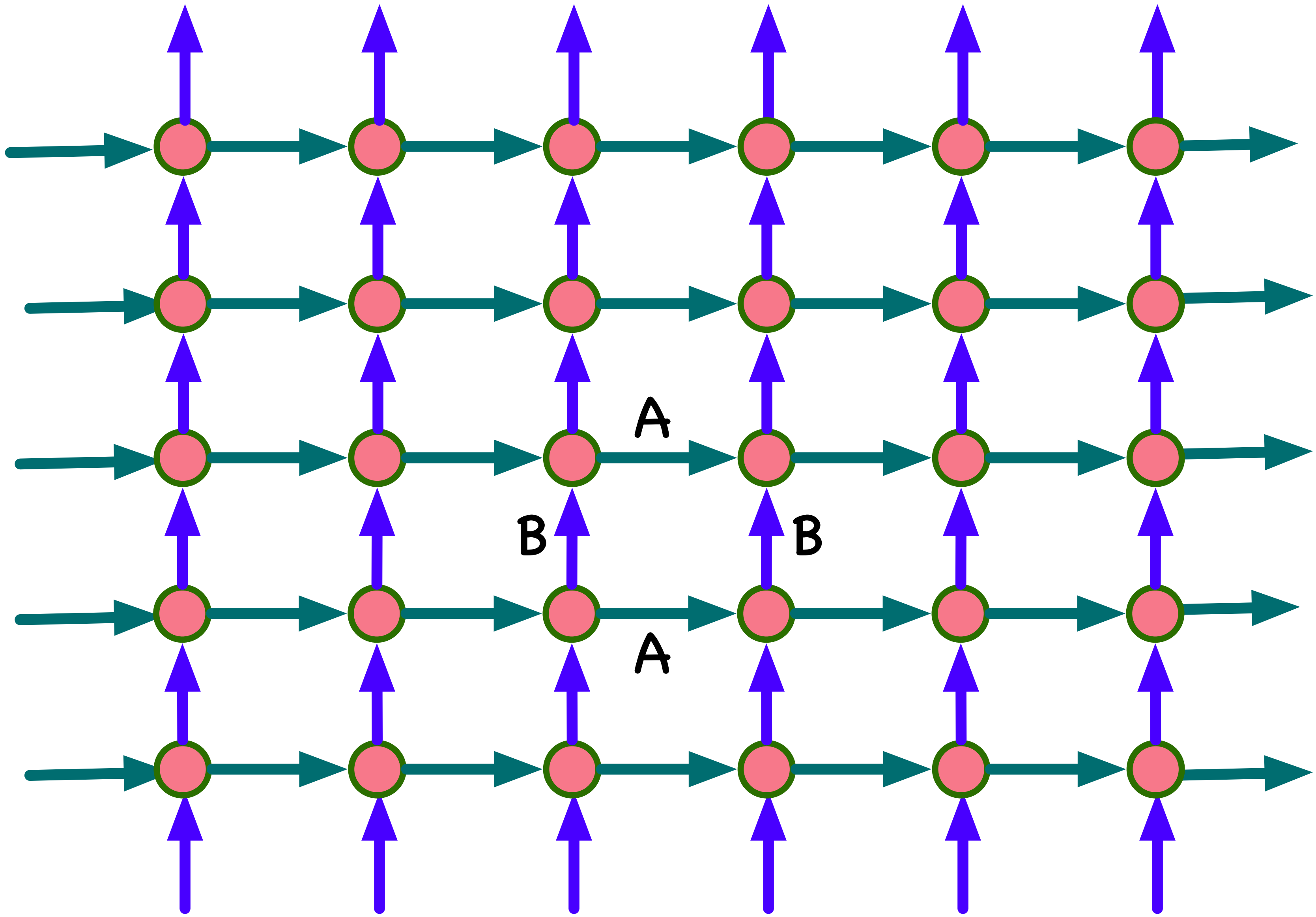
AAT = TA



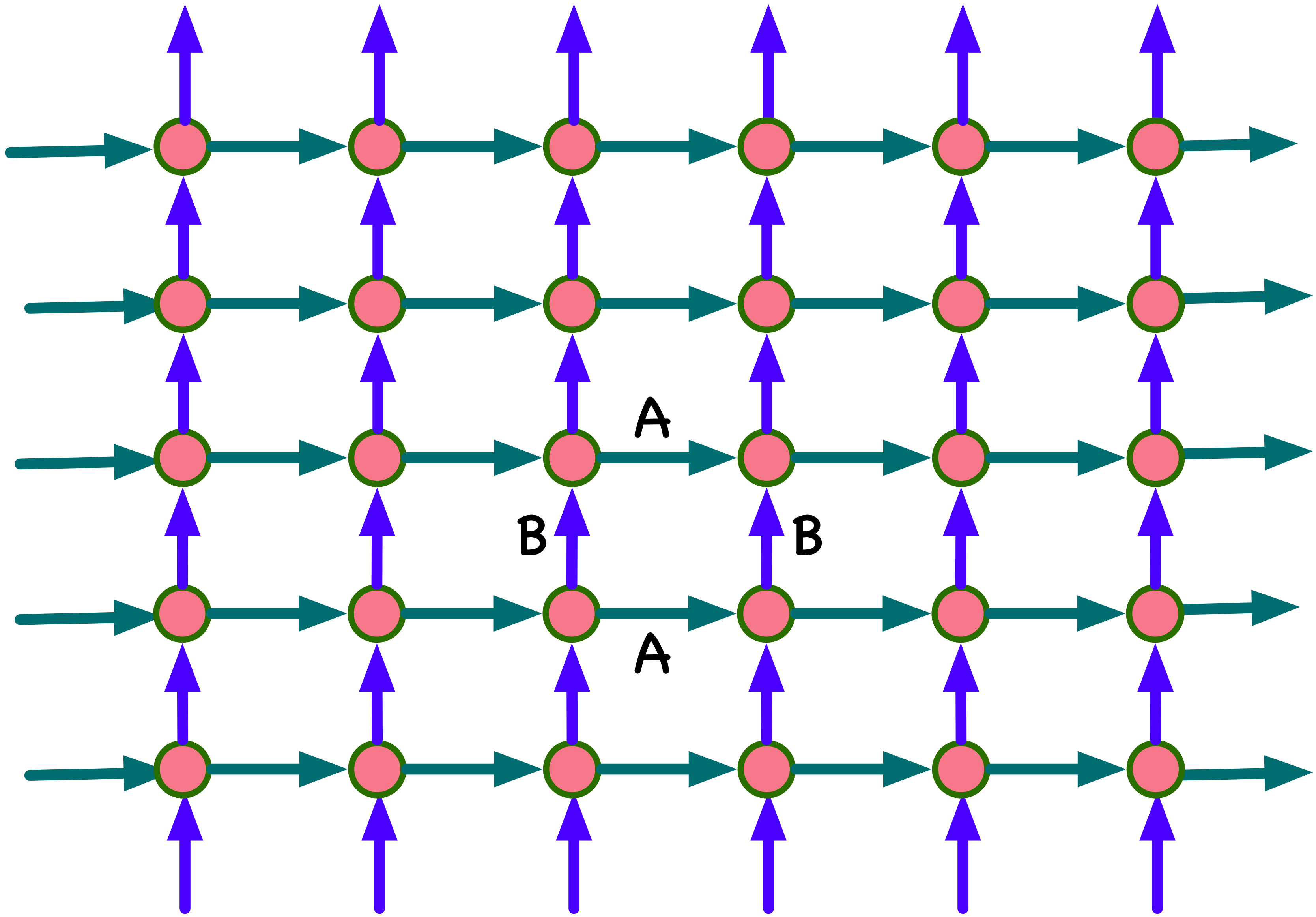
Here is Z^3, with generators

$A = (1,0,0)$, $B = (0,1,0)$, and $C = (0,0,1)$.

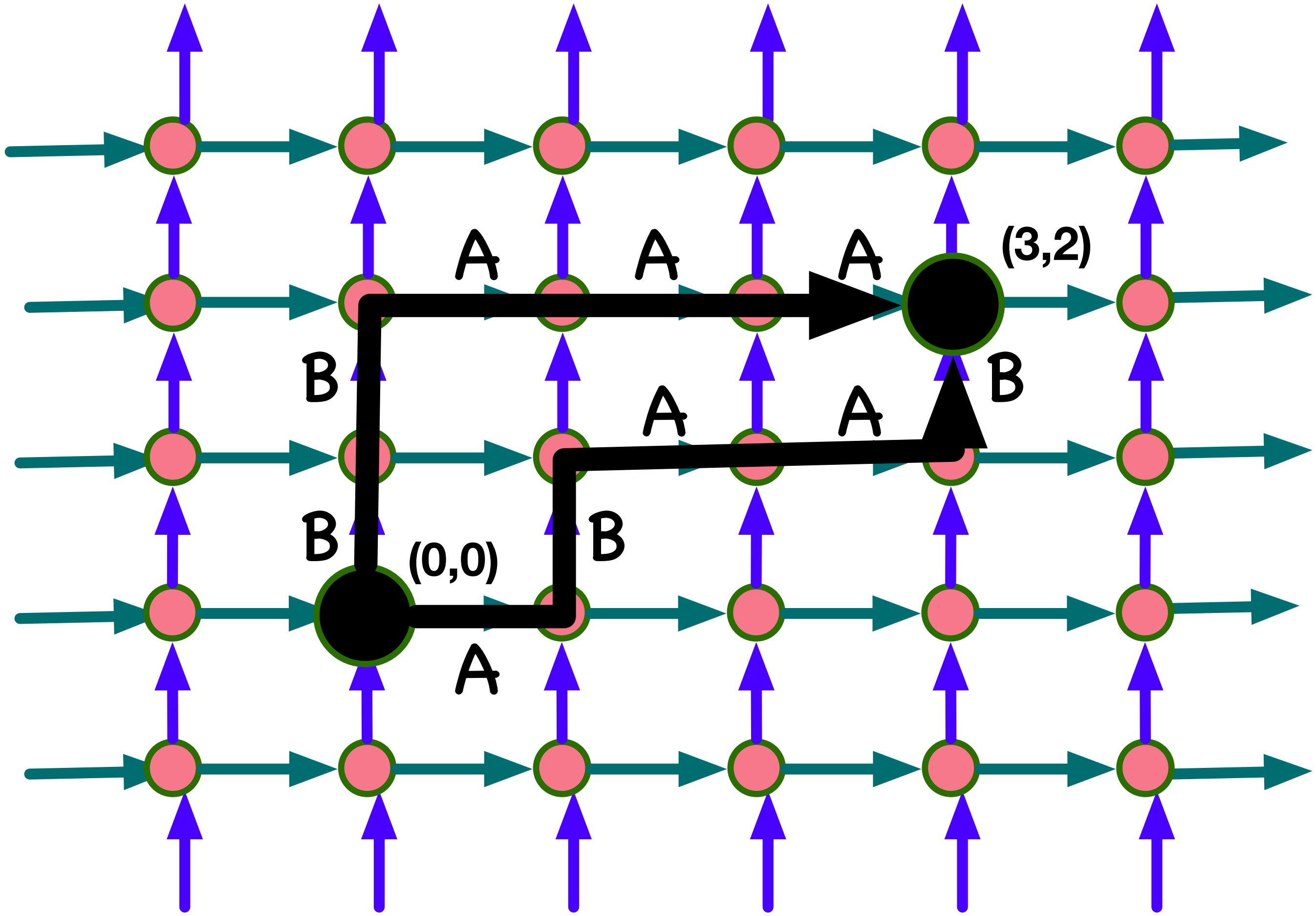
Geometry of Groups



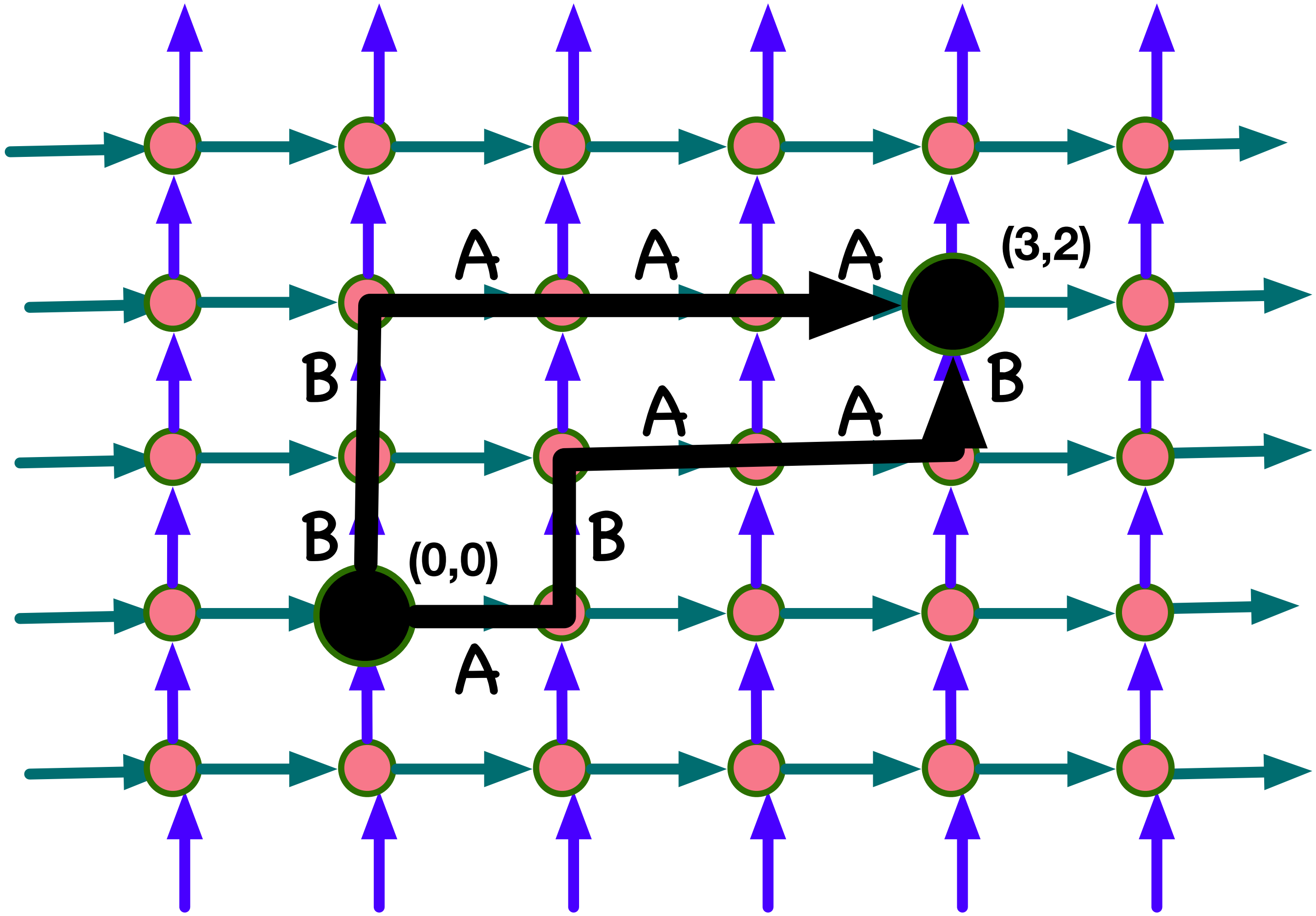
What are “straight lines”?



“Straight lines” are shortest paths.



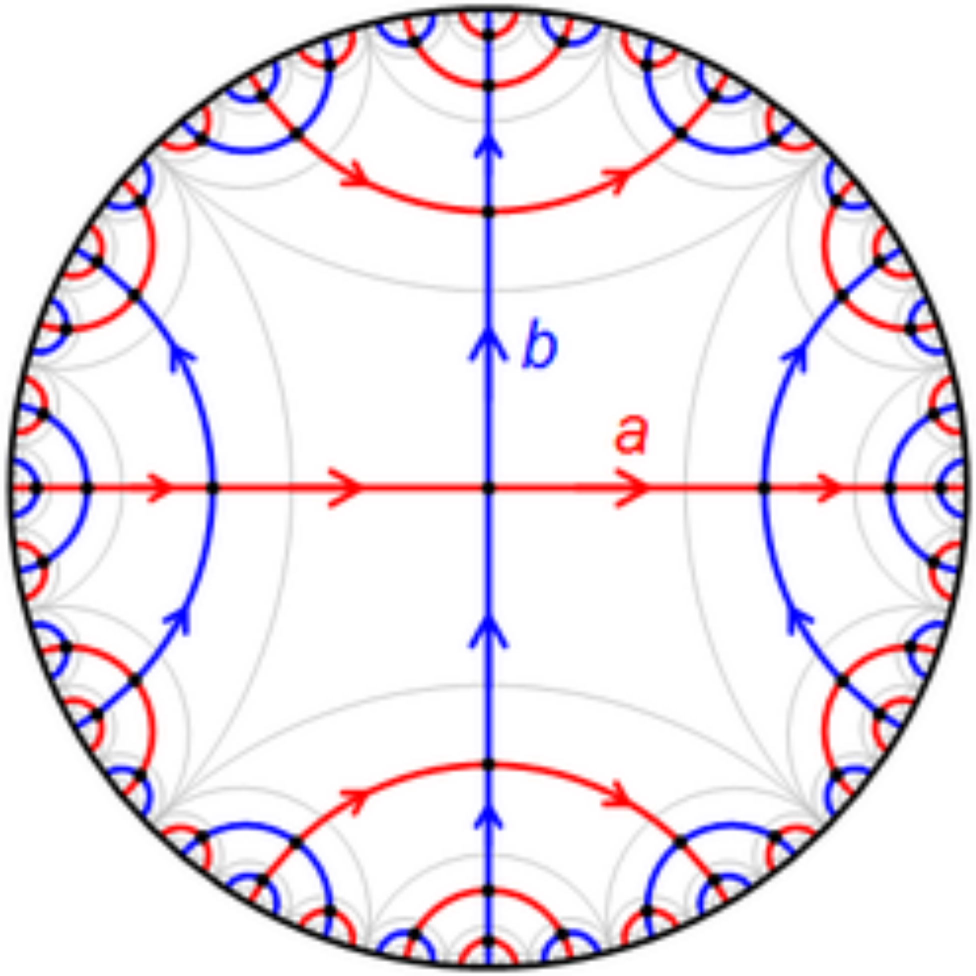
How many shortest ways are there to write the element AAABB?



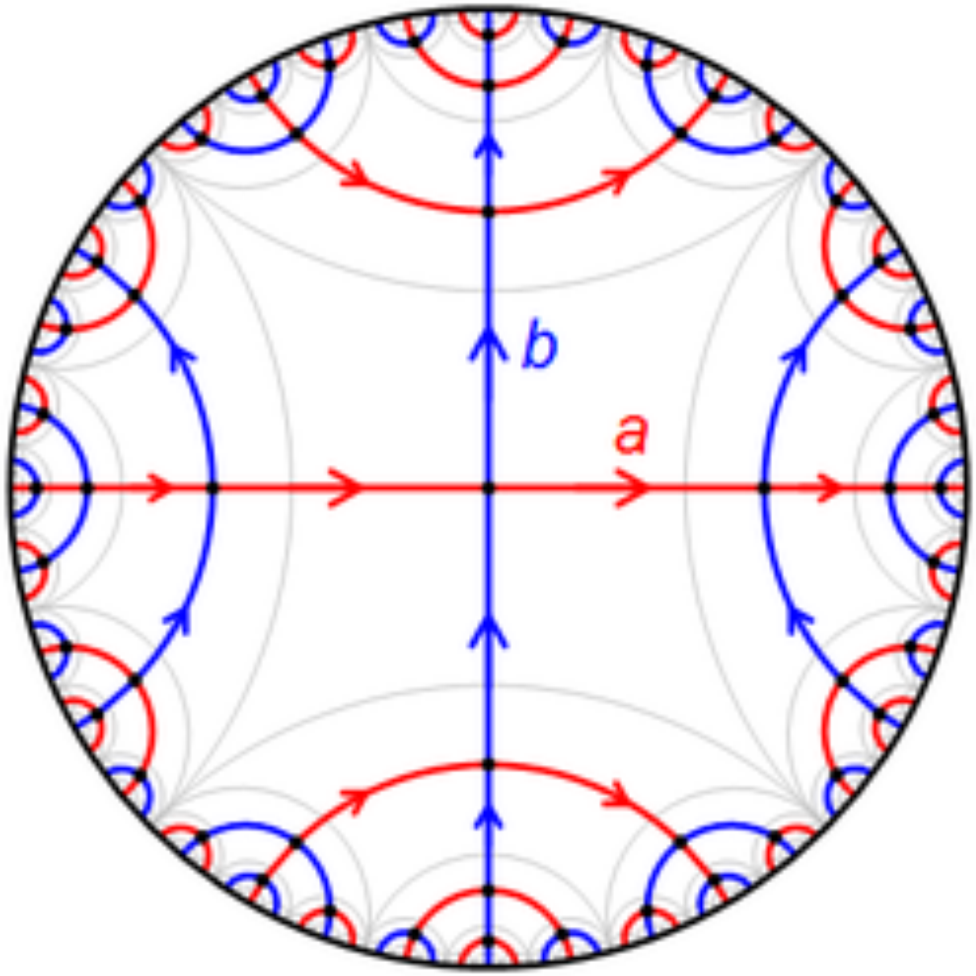
10 shortest ways to write this element:

BBAAA, BABAA, BAABA, BAAAB, ABBAA,

ABABA, ABAAB, AABBA, AABAB, AAABB

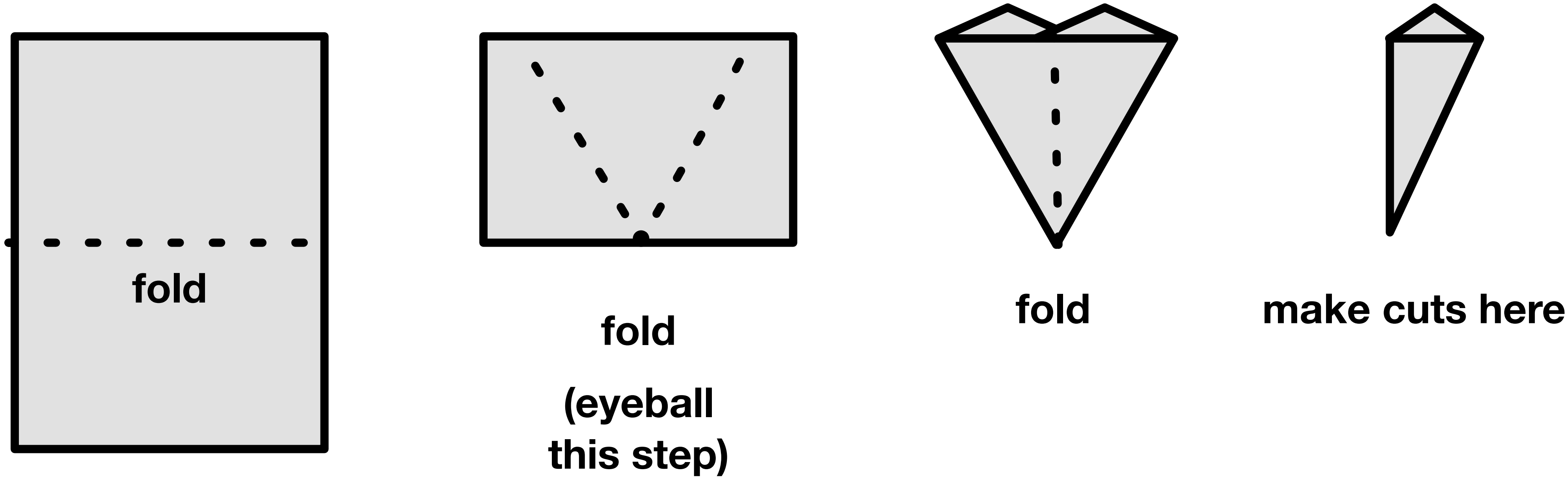


What geometry does this look like?



There is just one shortest way to write AAABB.

Puzzle: Draw the group of symmetries of a snowflake.



Instructions for making a snowflake.

شكرا جزيلا !

Some cool links

1. Vi Hart's doodle music:

https://www.youtube.com/watch?v=Av_Us6xHkUc

2. Group theory

http://www.math.uconn.edu/~kconrad/math216/whygroups.html

3. Symmetry groups:

https://www.youtube.com/watch?v=OHA6Hcj7P8o